SOME GENERAL ASPECTS OF MODERN GEOMETRY.*

BY PROFESSOR E. J. WILCZYNSKI.

IT is a great honor and an exceptional privilege to be asked to address such a distinguished audience as is assembled here upon this occasion. And so my first duty is the simple and elementary one of expressing to the officers of the societies, meeting in joint session, my gratitude for having selected me for such a task. But the task itself is not a simple one. Unwelcome as it may be, the fact remains that the workers in the fields of mathematics, physics, and astronomy, intimate associates in former times, have become comparative strangers. So widely have their various dialects diverged from the common mother tongue, that they find it possible to follow each other's speech only when great care is taken to articulate distinctly, and even then only at the expense of most intense and rigid attention. But while we may find it difficult to understand each other, after all, these sister sciences have much in common. The love and respect which they bear each other are still alive. They appreciate fully how great are the services which they can render each other, and how fruitful are those domains of thought in which these various subjects are made to intermingle. It is well that we should specialize, for only by intense application of intellectual forces to specific problems can real progress in science be made. But, unless we preserve a broad interest in a larger field, we run into the danger of losing a proper sense of balance and perspective. It is not true, even in science, that all things are of equal value, and it is better for science that we should study important problems rather than unimportant ones. But which problems *are* important, and which are not? Here is a question which is worth some thought. We know that it cannot be answered from the utilitarian point of view, at least not in an adequate and permanent fashion. We also know that any attempt to impose upon each other our *individual* criterion of value can only

* Address delivered before a joint meeting of the American Mathematical Society, the American Physical Society, the Astronomical and Astrophysical Society of America, and Sections A and B of the American Association for the Advancement of Science. Cleveland ,Ohio. December 31, 1912.

result in harm. It is well then that we should meet and discuss our problems, that we should attempt to formulate into general principles the results of our daily meditations, in the hope that these principles, whose value has been tested in our own individual experience, may prove to be helpful in some other related field. This is the interpretation which I have placed upon my task, an attempt to show how one great unifying principle pervades the whole realm of geometry.

The distinction between analysis and geometry, while a convenient one, is really superficial and in some respects injurious. For, if there is any one thing which the invention of analytic geometry has taught us, it is this: that every problem of analysis is capable of a geometrical interpretation, and that every problem of geometry may be formulated analytically. It is my individual conviction that no mathematical investigation is truly complete unless it puts into evidence the existing relations both from an analytic and a geometric point of view. It is true that geometric intuition has occasionally led mathematicians into error, but the first intuitions of analysts have also frequently been found wanting. *All* of our intuitions must be subjected to rigorous criticism. Geometry obeys the same laws of logic as analysis, and the disrepute into which it has fallen, in some quarters, is due to the fact that naive geometric intuitions have been compared, to their disadvantage, with refined analytical theories rather than with the naive analytic intuitions to which they really correspond. But, aside from the question of rigor, it is very important that our mathematical theorems should present themselves to us, *not* merely as the final consequences of long and complicated arguments. They are not truly our own, we have not fully seized their significance, until from some point of view they appear to be obviously and intuitively true. In very many cases, geometry furnishes the best method for thus intuitively grasping the full import of a mathematical situation. And this is true, not merely in the case of rough and simple analogies, but also in those very cases in which an untrained and naive intuition had caused the earlier students to go astray. Thus, for instance, the theory of uniform convergence, as presented by Osgood in geometric form, assumes a convincing force which no mere analytic treatment could give it, although the logic is precisely the same whether the argument be presented analytically or geometrically.

There is then, to my mind, no fundamental distinction between geometry and analysis. If, nevertheless, I have used the word geometry in the title of this address, I have done so because, according to the traditional classification, the questions which I shall discuss are generally regarded as questions of geometry.

The peculiar efficacy of geometric notions for illustrating an abstract argument has given rise to a striking paradox. The elements of analytic geometry have taught us to associate with a point in a plane a pair of numbers, and with a point of space a system of three numbers, the coördinates of the point. The desire for complete parallelism between analysis and geometry has led to the notion of a point in a space of n dimensions as the geometric image of a system of n numbers $(x_1, \cdots, x_n)$. Although we surely cannot be said to have any primitive intuitive notions as to the properties of a space of n dimensions, we nevertheless speak of curves, surfaces, etc., in such a space, the analogies indicated by this geometric manner of speech being extremely valuable and suggestive for the purposes of analysis. Thus, and this is the paradox to which I am referring, we make use of the abstract idea of an n-space, of which we have no direct geometric intuition, to render intelligible so concrete a thing as a system of n numbers.

The idea of a space of n dimensions (or an n-space) has now become an essential part of our mathematical patrimony. The notion developed gradually, and traces of it are to be found in the writings of several of the mathematicians of the latter part of the eighteenth and the early part of the nineteenth century, especially in those of Gauss and Cauchy. The complete notion of an n-space, however, with all of its most essential implications, must be ascribed to Grassmann, whose first "Ausdehnungslehre" of 1844 is largely devoted to this subject.

It seems, at first thought, as though the dimensionality of a space ought to be regarded as its most important characteristic, and that it would be vain to attempt to look at the same space in two different fashions, so as to attribute to it two different dimensionalities. It seems, also, as though nothing could be more hopeless than to attempt to associate a genuine geometric intuition with such an abstract notion as that of an n-space. And indeed, if all of our thinking were abstract, I doubt whether the possibility of doing either

of these things would ever have occurred to anybody. From another point of view, however, which gradually presented itself in the course of development of mathematical thought, the affirmative answers to both of these questions become so evident as to appear almost trivial. Two geometers of the early part of the nineteenth century, Poncelet and Gergonne, had discovered what is now known as the principle of duality, according to which every theorem of projective geometry may be made to yield a second one by the simple process of interchanging the words point and plane, and leaving the word line unchanged. It thus became apparent that, for the purposes of projective geometry, the point and the plane were coördinate notions. To the already existing ideas of curves and surfaces thought of as point loci, were added the strictly correlative notions of one and two-dimensional aggregates of planes and their respective envelopes. Thus, as a consequence of the principle of duality, for the first time an element different from a point, namely a plane, was thought of as the generating unit of geometric forms. But our ordinary space is three-dimensional from the point of view of its planes as well as of its points, so that the dimensionality of space was still left unchanged. Moreover, this single instance of a change of the space element was too isolated and special a thing to inspire any easy or far-going generalization. It was a great step in advance, therefore, when Plücker in 1846 proposed to regard the straight line as the generating element of space, and introduced the notion of line coördinates. For here, for the first time, do we find space presenting itself as a four-dimensional aggregate, thus destroying the idea of dimensionality as an inherent geometric characteristic of space. And here too, do we find the fountain head for all of those generalizations of modern geometry, in which not merely the point, plane, and line, but countless other geometric forms appear as generating elements. There is nothing easier, nowadays, than to represent concretely in the plane, a geometry of any number of dimensions.

For the purpose of characterizing some particular branch of geometric research, the choice of the space element is particularly important. The same analytic theorems may, by a change of space element, receive many widely differing geometrical interpretations. The principle of duality, to which I have already alluded, is probably the best known illustration

of this fact. Another system of abstractly equivalent geometric theories is given by Plücker's line geometry, Lie's sphere geometry, and the geometry of a quadric four-spread in a space of five dimensions. Similarly it is, abstractly speaking, the same thing whether we are discussing a linear space of five dimensions, the aggregate of conics in a plane, or the totality of linear complexes in ordinary space. But the geometric content of our theorems is very different in these various cases. It is not necessarily desirable, even in any particular investigation, to consider always the same geometric form as space element. In fact, one of the most fruitful results of the discussions of Plücker and his successors is the freedom which they have given to the present day geometer to change his space element whenever the change may seem desirable.

Let us suppose that we have selected some geometric form as generating element for a particular geometric theory which we wish to develop. If this element requires k numbers for its complete determination we may, in accordance with our previous remarks, speak of it as a "*point*" in a space of k dimensions, this *space* being the aggregate of all such elements. This *space* may have the property that every point of the line which joins any two of its points itself belongs to this space.* If this is so we shall call it a linear space. Such is, for instance, the two-dimensional aggregate of all of the points of a plane, the three-dimensional aggregate of all points of ordinary space, the five-dimensional aggregate of all conics in a plane. As an illustration of a non-linear two-dimensional space we may take the aggregate of all of the points of a curved surface. Such a curved surface may, however, be regarded as immersed in a linear space of three dimensions, and indeed this is our customary way of looking at it. In the same way, if the space of k dimensions determined by our space element is not a linear space, we shall think of it as immersed in a linear space of $n > k$ dimensions, choosing the number n as small as may be compatible with the nature of our original non-linear k-space. Now a point of this linear n-space is determined by n coördinates $x_1, \cdots, x_n$. But since the aggregate of all of the geometric forms which we are using as space elements has only k dimensions we shall have to think of $x_1, \cdots, x_n$ as

* Of course the application of this criterion presupposes a definition of the word line. But we cannot discuss such details in this address.

satisfying $n - k$ independent equations none of which are of the first degree, since otherwise our k-dimensional aggregate of space elements would belong to a linear space of less than n dimensions. In most applications these $n - k$ equations are algebraic.

In the language of hypergeometry we are then dealing with a point upon an algebraic k-spread immersed in a space of n dimensions. This k-spread is characterized by the $n-k$ independent algebraic equations

$$(1) \qquad f_i(x_1, x_2, \cdots, x_n) = 0 \quad (i = 1, 2, \cdots, n - k),$$

none of which is of the first degree.*

Let us consider, first, the case that no such equations are present, so that all of the points of our n-space are available as generating elements for the geometric forms which we wish to study. Any system of values $(x_1, \cdots, x_n)$ gives us a "point" of such a space; several such systems give us several "points." We are primarily interested in the case where we have an infinite number of such points. If we assume that the points of such an infinite set form a continuous analytic aggregate, we shall have expressions of the form

$$(2) \qquad x_1 = \varphi_1(u_1, \cdots, u_r), \ \cdots, \ x_n = \varphi_n(u_1, \cdots, u_r)$$

for their coördinates, where r may be any integer between 1 and n, and where $\varphi_1, \cdots, \varphi_n$ are analytic functions of their arguments. If $r = 1$ we have a one-dimensional spread, or curve, composed of a single infinity of points of our n-space S_n. If $n = 2$ we have a two-dimensional spread, or surface, immersed in S_n. In every case we find an analytic r-spread immersed in our space S_n of n dimensions, which for $r = n$ coincides with S_n itself, or at least with an n-dimensional portion of S_n.

If we use three coördinates for a point in ordinary space, some complications arise, caused by the exceptional rôle played by the "points at infinity." To avoid this difficulty it has long been customary to introduce the so-called homogeneous coördinates. For precisely the same reason it will be advantageous to introduce homogeneous coördinates for the points of our n-space. Let us write

* If the k-spread is to be irreducible, it may require more than $n - k$ equations to characterize it completely.

$$x_1 = \frac{y_1}{y_{n+1}}, \quad x_2 = \frac{y_2}{y_{n+1}}, \quad \cdots, \quad x_n = \frac{y_n}{y_{n+1}},$$

i. e., let us introduce a system of $n+1$ numbers $y_1, y_2, \cdots, y_{n+1}$ whose ratios are equal to $x_1, x_2, \cdots, x_n$ respectively. These $n+1$ numbers, only whose ratios are of interest for us, are called the homogeneous coördinates of the point. The homogeneous coördinates of any point of our r-spread will then be given by $n+1$ equations of the form

$$y_1 = \psi_1(u_1, \cdots, u_r), \quad y_2 = \psi_2(u_1, \cdots, u_r), \quad \cdots, \quad y_{n+1} = \psi_{n+1}(u_1, \cdots, u_r), \tag{3}$$

the geometrical content of which equations would not be altered if we were to multiply all of their right members by any common factor $\lambda(u_1, \cdots, u_r)$, since such a multiplication obviously has no influence upon the values of the ratios $y_1 : y_2 : \cdots : y_{n+1}$. We may assume, of course, that the functions $y_1, \cdots, y_{n+1}$ are linearly independent. For, if they were not, we could reduce our problem to a similar one in a linear space of fewer than n dimensions.

We wish to show that we can always find a system of linear homogeneous differential equations of which $y_1, y_2, \cdots, y_{n+1}$ are the fundamental solutions, in the sense that the most general solution of the system will have the form

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_{n+1} y_{n+1}, \tag{4}$$

where $c_1, c_2, \cdots, c_{n+1}$ are arbitrary constants.

The truth of this statement is obvious for $r = 1$. We are then dealing with a *curve* of S_n, and $y_1, y_2, \cdots, y_{n+1}$ may be regarded as the fundamental solutions of an ordinary linear homogeneous differential equation of the $(n+1)$th order

$$\frac{d^{n+1}y}{du^{n+1}} + p_n(u)\frac{d^n y}{du^n} + \cdots + p_1(u)\frac{dy}{du} + p_0(u)y = 0, \tag{5}$$

whose general solution will then be given by (4).

If $r > 1$ we shall have to consider partial differential equations. A function of r independent variables has r partial derivatives of the first order, $\frac{1}{2}r(r+1)$ partial derivatives of the second order, etc., $[r(r+1)\cdots(r+k-1)]/k!$ partial

derivatives of order k. We may think of the function itself as its zeroth derivative. Thus there will be altogether

$$p_k = 1 + \sum_{i=1}^{k} \frac{r(r+1)\cdots(r+i-1)}{i!} \tag{6}$$

partial derivatives of a function y of $u_1, \cdots, u_r$, whose order does not exceed k. This number grows very rapidly with k, and we may obviously choose k so large that p_k shall become greater than $n+1$, n being the number of dimensions of the space under consideration.

If our r-spread does not degenerate into an $r-1$-spread, $y_1, \cdots, y_{n+1}$ cannot satisfy one and the same linear homogeneous partial differential equation of the first order

$$Ay + B_1 \frac{\partial y}{\partial u_1} + \cdots + B_r \frac{\partial y}{\partial u_r} = 0.$$

For, if they did, the ratios of $y_1, y_2, \cdots, y_{n+1}$ would be functions of at most $r-1$ combinations of $u_1, \cdots, u_r$, i. e., we should be at most dealing with an $(r-1)$-spread.

All of the functions $y_1, \cdots, y_{n+1}$ may, however, satisfy one or several such partial differential equations of the second order. If they satisfy as many as $\frac{1}{2}r(r+1)$ independent equations of this kind, *all* of the second order derivatives can be expressed in the form

$$\frac{\partial^2 y}{\partial u_i \partial u_k} = A_{ik}y + B_{i,\,k}{}^{(1)} \frac{\partial y}{\partial u_1} + \cdots + B_{i,\,k}{}^{(r)} \frac{\partial y}{\partial u_r}.$$
$$(i, k = 1, 2, \cdots, r),$$

and therefore also all of the derivatives of higher order. The most general analytic solution of such a system is clearly a linear homogeneous combination with constant coefficients of $r+1$ independent ones, so that our r-spread must be contained in a linear r-space. Since we have assumed that the n-space, which we have under consideration, is the linear space of lowest dimensionality which contains our r-spread, this case can present itself only if $r = n$.

In general, our $n+1$ functions $y_1, \cdots, y_{n+1}$ will satisfy fewer than $\frac{1}{2}r(r+1)$ linear homogeneous partial differential equations of the second order, perhaps none at all. If the number of such equations is $\frac{1}{2}r(r+1) - s$, we may regard

y, $\partial y/\partial u_1$, $\cdots$, $\partial y/\partial u_r$, and s of the second order derivatives as linearly independent while the remaining second order derivatives are expressible linearly and homogeneously in terms of these $1+r+s$ quantities with coefficients which may be functions of $u_1, \cdots, u_r$. If $1+r+s$ is less than $n+1$, we examine the derivatives of the third order. Suppose that all of these are expressible linearly and homogeneously in terms of the above $1+r+s$ quantities and of t independent third order derivatives. Let us continue this process. We shall finally have all of the partial derivatives of a certain, say the kth, order expressed linearly and homogeneously in terms of $1+r+s+\cdots+w$ of them, where k is so large that for the first time

$$1+r+s+\cdots+w \geqq n+1. \tag{7}$$

For, if this were not so, we could express all of the partial derivatives, of all orders, linearly and homogeneously in terms of less than $n+1$ independent ones and our r-spread would be contained in a linear space of fewer than n dimensions, contrary to our hypothesis.

But the sum $1+r+s+\cdots+w$ cannot exceed $n+1$. For, if it did, our r-spread could not be contained in any linear n-space. Therefore we have

$$1+r+s+\cdots+w = n+1. \tag{8}$$

We have found a system of linear homogeneous differential equations, consisting of $\frac{1}{2}r(r+1)-s$ equations of the second order, $\frac{1}{3}r(r+1)(r+2)-t$ equations of the third order, etc., $[r(r+1)\cdots(r+k-1)]/k!-w$ equations of the kth order. In most cases, these equations and those obtained from them by differentiation will enable us to express all of the derivatives of order $k+1$ linearly and homogeneously in terms of the $n+1$ independent ones of lower order, and the same thing will then be true of all derivatives of order $k+2$, $k+3$, etc. If, however, some of the derivatives of order $k+1$ cannot be determined in this way, we can always add to our system a sufficient number of equations of order $k+1$ of which y_1, $\cdots$, y_{n+1} will also be solutions, to insure that all derivatives of order $k+1$ will appear as linear homogeneous functions of the $n+1$ fundamental derivatives of lower order.

The coefficients of this system will be analytic functions of

$u_1, \cdots, u_r$ and its most general solution will be of the form

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_{n+1} y_{n+1}, \tag{9}$$

where $c_1, \cdots c_{n+1}$ are arbitrary constants.

Such a system of partial differential equations is called a completely integrable system.

Obviously, if any completely integrable system is given, its solutions may be interpreted as the coördinates of the points of an r-spread in n dimensions. We see, therefore, that to every analytic r-spread contained in a linear space of n dimensions, there corresponds a completely integrable system of linear homogeneous partial differential equations whose general solution contains $n+1$ arbitrary constants, and conversely.

But we can give a more precise significance to our result. The r-spread with which we started is not the only one which satisfies our completely integrable system of equations. The r-spread, whose equations are

$$\begin{aligned} \bar{y}_1 &= c_{11} y_1 + c_{12} y_2 + \cdots + c_{1,\,n+1} y_{n+1}, \\ &\cdots\cdots\cdots\cdots\cdots \\ \bar{y}_{n+1} &= c_{n+1,\,1} y_1 + c_{n+1,\,2} y_2 + \cdots + c_{n+1,\,n+1} y_{n+1}, \end{aligned} \tag{10}$$

where the quantities c_{ik} are arbitrary constants with a non-vanishing determinant, will also satisfy the same system of partial differential equations. Moreover, since we know that (9) is the expression for the most general solution of our system, no r-spreads, other than those expressible by (10), will satisfy the same system of partial differential equations. Now the equations (10) are precisely the equations of the most general projective transformation of our n-space, projective transformations being those which convert every linear k-spread of the space again into a linear k-spread. Thus, by means of our system of partial differential equations alone, we shall not be able to distinguish between the original r-spread and any one of its projective transformations. The properties which are common to all of these projectively equivalent r-spreads are called projective properties. Consequently our completely integrable system of equations, taken by itself, is concerned only with the projective properties of the r-spread.

However, the analytical representation of our r-spread

given by equations (3), and consequently the resulting system of partial differential equations, contains some elements which cannot fairly be said to belong to the r-spread itself, and which may be changed without giving rise to a corresponding change in the r-spread. In fact, as we have already noticed, we may multiply $y_1, \cdots, y_{n+1}$ by a common factor $\lambda(u_1, \cdots, u_r)$ without changing the r-spread, since $y_1, \cdots, y_{n+1}$ are homogeneous coördinates. Furthermore an arbitrary transformation of the form

$$v_k = \varphi_k(u_1, u_2, \cdots, u_r) \qquad (k = 1, 2, \cdots, r)$$

merely changes the parameters to which the r-spread is referred, without affecting the r-spread itself. We are thus led to transform our system of partial differential equations by the transformations

$$(T) \qquad \begin{aligned} \eta &= \lambda(u_1, \cdots, u_r)y, \\ v_k &= \varphi_k(u_1, \cdots, u_r) \quad (k = 1, 2, \cdots, r), \end{aligned}$$

where the functions $\lambda, \varphi_1, \cdots, \varphi_r$ are arbitrary functions of their arguments. All of the systems obtained in this way from a given one correspond to the same class of projectively equivalent r-spreads. Those combinations of the coefficients and of the variables of our system of partial differential equations which are left unchanged when we make any transformation of the form (T) are called its *invariants* and *covariants*. Their values give the true and adequate expression of the projective properties of the r-spread, in a form independent of the accidental elements of any particular analytic representation.

We have discussed, so far, the case that no equations of the form (1), initially limiting us to the points of an algebraic k-spread of our n-space, are present. But it makes little difference for our theory if such equations do appear. Since our r-spread must then be contained in the k-spread whose equations are given by (1), the functions $y_1, \cdots, y_{n+1}$ will of their own accord satisfy these equations. If, however, instead of starting out with the explicit equations of a given r-spread, we were to begin our theory with a given completely integrable system of partial differential equations, we should have to impose upon its solutions the condition of satisfying the conditions (1). This may be done, very simply, by adding these equations (1) as subsidiary conditions to our system. It

may also be done, but this may involve greater difficulties, by imposing appropriate conditions on the coefficients of the system.

Thus, *the projective geometry of an analytic r-spread in a linear space of n dimensions is equivalent to the theory of the invariants and covariants of a completely integrable system of linear partial differential equations with r independent variables, whose general solution depends on n + 1 arbitrary constants.*

If we recall our preliminary discussion regarding the arbitrariness of the space element, and the great generality which is therefore involved in the notion "*r*-spread in *n* dimensions" even as applied to ordinary space, we shall appreciate the sweeping character of this generalization which unifies such a vast domain. To the mathematician who knows that metric properties may, in a certain sense, be regarded as projective properties, it will be evident what must be added in order that this unifying principle may embrace metric geometry as well.

THE UNIVERSITY OF CHICAGO,
December, 1912.

ON CERTAIN NON-LINEAR INTEGRAL EQUATIONS

BY MR. H. GALAJIKIAN.

(Read before the American Mathematical Society, December 31, 1912.)

NON-LINEAR integral equations of the Volterra type have been considered by Lalesco,* Cotton,† and Picone.‡ The two theorems of the present paper give results which are of more general character. Theorems apparently still more general have been stated very recently by Evans.§ The method used is that of successive approximations. The plan of treatment applies to integral equations of the type

* *Journal de Mathématiques*, series 6, vol. 4 (1908), p. 165; Introduction à la Théorie des Equations Intégrales, p. 127.

† *Bulletin de la Société math. de France*, vol. 38 (1910), p. 144.

‡ *Rendiconti del Circolo Matem. di Palermo*, vol. 30 (1910), p. 351.

§ Proceedings of the International Congress of Mathematicians, Cambridge, December, 1912. The present paper was completed without knowledge of Professor Evans' work, and forms one section of a Cornell University master's thesis, which was officially approved in May, 1912.

$$y(x) = g\left\{x, \int_{x_0}^{x} f_1[x, t, y(t)]dt, \cdots, \int_{x_0}^{x} f_m[x, t, y(t)]dt\right\},$$

or to a system of n such equations with n unknown functions; as a very special case we have the usual existence theorems for systems of ordinary differential equations. For brevity we consider, as a typical case, the integral equation

$$(1) \quad y(x) = g\left\{x, \int_{x_0}^{x} f_1[x, t, y(t)]dt, \int_{x_0}^{x} f_2[x, t, y(t)]dt\right\}.$$

THEOREM I. *Let* $g\{x, u_1, u_2\}$ *satisfy a Lipschitz condition*

$$|g\{x+\xi, u_1+\mu_1, u_2+\mu_2\} - g\{x, u_1, u_2\}| \leqq A\{|\xi| + |\mu_1| + |\mu_2|\},$$

in the region $|x - x_0| \leqq \alpha$, $|u_1| \leqq \beta$, $|u_2| \leqq \beta$. *Let* $f_1(x, t, y)$ *and* $f_2(x, t, y)$ *be continuous, and satisfy a Lipschitz condition for the argument* y

$$|f_i(x, t, y+\eta) - f_i(x, t, y)| \leqq B|\eta| \qquad (i = 1, 2),$$

in the region $|x - x_0| \leqq \alpha$, $|t - x_0| \leqq \alpha$, $|y - g\{x_0, 0, 0\}| \leqq \gamma$; *furthermore let*

$$|f_i(x, t, y)| \leqq M \qquad (i = 1, 2).$$

Then if ρ *satisfies the conditions*

$$\rho \leqq \alpha, \quad \rho \leqq \frac{\beta}{M}, \quad \rho \leqq \frac{\gamma}{A(1+2M)},$$

there exists, in the interval $|x - x_0| \leqq \rho$, *one and only one continuous solution* $y(x)$ *of the integral equation* (1).

Define the functions

$$y_0(x) = g\{x_0, 0, 0\}$$

$$(2) \quad y_n(x) = g\left\{x, \int_{x_0}^{x} f_1[x, t, y_{n-1}(t)]dt, \int_{x_0}^{x} f_2[x, t, y_{n-1}(t)]dt\right\} \qquad (n = 1, 2, \cdots).$$

At each stage of the approximation, we see that the function

$y_n(x)$ satisfies the conditions

$$|y_n(x) - g\{x_0, 0, 0\}| \leqq \gamma,$$

$$\left|\int_{x_0}^{x} f_i[x, t, y_n(t)]dt\right| \leqq \beta \qquad (i = 1, 2),$$

when $|x - x_0| \leqq \rho$; hence the next following approximation will have a meaning. Evidently each $y_n(x)$ is continuous, $|x - x_0| \leqq \rho$.

We shall prove that the sequence $y_n(x)$ approaches a limit uniformly. To this end write

$$y_n(x) - y_{n-1}(x) = Y_n(x) \qquad (n = 1, 2, \cdots),$$

so that

$$y_n(x) = y_0(x) + Y_1(x) + Y_2(x) + \cdots + Y_n(x) \qquad (n = 1, 2, \cdots).$$

The usual methods suffice to show that

$$|Y_n(x)| \leqq \frac{2^{n-1}A^nB^{n-1}(1 + 2M)|x - x_0|^n}{n!},$$

hence that

$$|Y_n(x)| \leqq \frac{2^{n-1}A^nB^{n-1}(1 + 2M)\rho^n}{n!} = \frac{1 + 2M}{2B}\frac{(2AB\rho)^n}{n!}$$

when $|x - x_0| \leqq \rho$. Since the series of constants

$$\sum_{n=1}^{\infty} \frac{(2AB\rho)^n}{n!}$$

converges, we see that the series $y_0(x) + Y_1(x) + \cdots + Y_n(x) + \cdots$ converges uniformly, $|x - x_0| \leqq \rho$; if we represent its value by $y(x)$ we have

$$\lim_{n=\infty} y_n(x) = y(x)$$

uniformly, $|x - x_0| \leqq \rho$. Thus $y(x)$ is a continuous function which, as we see by reference to (2), satisfies the integral equation (1).

That there is only one such continuous solution may be seen as follows. Suppose there were two, $y(x)$ and $z(x)$, and

put $y(x) - z(x) = w(x)$. Then if we write equation (1) for $y(x)$ and $z(x)$, and subtract, we see readily from the given conditions on the functions g, f_1, f_2, that

$$|w(x)| \leqq 2AB\left|\int_{x_0}^{x} |w(t)|dt\right|.$$

If now we write $|w(x)| \leqq W$, we find by successive applications of the preceding formula

$$|w(x)| \leqq 2ABW|x - x_0|,$$

$$|w(x)| \leqq \frac{4A^2B^2W|x - x_0|^2}{2!},$$

and in general,

$$|w(x)| \leqq \frac{W(2AB|x - x_0|)^n}{n!}.$$

Since the expression on the right has the limit zero as n becomes infinite, we see that $w(x) = 0$; thus $y(x) = z(x)$.

It is interesting to note that if we replace the condition

$$|y(x) - g\{x_0, 0, 0\}| \leqq \gamma$$

on the arguments of f_1, f_2, by the condition

$$|y(x) - g\{x, 0, 0\}| \leqq \gamma,$$

we may broaden the results of the theorem in two ways: first, in that we impose on the function $g\{x, u_1, u_2\}$ a Lipschitz condition involving only the last two arguments, instead of all three; secondly, in that we give the solution in a less restricted interval. The facts are stated as follows:

THEOREM II. *Let $g\{x, u_1, u_2\}$ be continuous, and satisfy a Lipschitz condition for the arguments u_1, u_2*

$$|g\{x, u_1 + \mu_1, u_2 + \mu_2\} - g\{x, u_1, u_2\}| \leqq A\{|\mu_1| + |\mu_2|\}$$

in the region $|x - x_0| \leqq \alpha$, $|u_1| \leqq \beta$, $|u_2| \leqq \beta$. Let $f_1(x, t, y)$ and $f_2(x, t, y)$ be continuous, and satisfy a Lipschitz condition for the argument y

$$|f_i(x, t, y + \eta) - f_i(x, t, y)| \leqq B|\eta| \qquad (i = 1, 2)$$

in the region $|x - x_0| \leqq \alpha$, $|t - x_0| \leqq \alpha$, $|y - g\{x, 0, 0\}| \leqq \gamma$;

furthermore let

$$|f_i(x, t, y)| \leqq M \qquad (i = 1, 2).$$

Then if ρ *satisfies the conditions*

$$\rho \leqq \alpha, \quad \rho \leqq \frac{\beta}{M}, \quad \rho \leqq \frac{\gamma}{2AM},$$

there exists, in the interval $|x - x_0| \leqq \rho$, *one and only one continuous solution* $y(x)$ *of the integral equation* (1).

The proof is entirely similar to that of Theorem I.

CORNELL UNIVERSITY,
January 10, 1913.

A THEOREM ON ASYMPTOTIC SERIES.

BY MR. VINCENT C. POOR.

THEOREM: *If* $f(z)$ *is not holomorphic at* $z = 0$, *but is formally developable into a Maclaurin series, and if* w *is asymptotic to* $a_1/z + a_2/z^2 + \cdots$ (*written:* $w \sim a_1/z + a_2/z^2 + \cdots$), *then* $f(w)$ *has an asymptotic representation.**

To prove this theorem take $f(z)$ in the form

$$f(z) = f(0) + f'(0)z + \frac{f''(0)}{2!}z^2 + \cdots + \frac{f^{(n)}(0)z^n}{n!} + \int_0^z f^{(n+1)}(t) \cdot \frac{(z-t)^n}{n!}\,dt. \tag{1}$$

Since

$$w \sim \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots$$

w may be written

$$w = \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots + \frac{a_n + \epsilon_{1nz}}{z^n}, \tag{2}$$

where, according to the Poincaré definition† for an asymptotic

* This theorem is the "résultat préalablement obtenu" referred to in Professor Ford's paper in the *Bulletin* of the French Society for 1911. See *Bulletin Société math. de France*, vol. 40 (1912), fascicule 1 under "Erratum du Tome XXXIX."

† Poincaré, *Acta Mathematica*, vol. 8 (1886), p. 296.

series,

$$\lim_{z=\infty} \epsilon_{1nz} = 0.$$

Replacing z by w in (1) and collecting coefficients of like powers, the following equation results:

$$(3) \qquad f(w) = f(0) + \frac{A_1}{z} + \frac{A_2}{z^2} + \cdots + \frac{A_n + \epsilon_z}{z^n} + R,$$

where

$$R = \int_0^w f^{(n+1)}(t) \cdot \frac{(w-t)^n}{n!}\, dt$$

and where the A_i $(i = 1, \cdots, n)$ are readily determined from the substitution.

$$\epsilon_z = \epsilon_{1nz} + \epsilon_{2nz} + \cdots + \epsilon_{nnz}$$

and

$$\epsilon_{inz}/z^n \qquad (i = 1, \cdots, n).$$

are the remainders after n terms of w, w^2, w^3, $\cdots$, w^n, respectively. Since

$$\lim_{z=\infty} \epsilon_{inz} = 0 \qquad (i = 1, \cdots, n),$$

it follows that

$$\lim_{z=\infty} \epsilon_z = 0.$$

To satisfy the Poincaré definition of an asymptotic series, it remains to show that

$$\lim_{z=\infty} z^n R = 0.$$

For z real this last condition may be shown to hold as follows: $(w-t)^n$ does not change sign in the interval $0 \leqq t \leqq w$, therefore the first law of the mean for integrals may be applied. Making this application, it thus obtains that

$$(4) \quad z^n R \equiv \frac{z^n}{n!}\int_0^w f^{(n+1)}(t) \cdot (w-t)^n dt = \frac{z^n \theta_z M}{n!}\int_0^w (w-t)^n dt,$$

where M is the maximum value of $|f^{(n+1)}(t)|$ in the interval $0 \leqq t \leqq w$, and where $-1 \leqq \theta_z \leqq 1$. Evaluating the integral

in (4) and taking the limit, it is found that

$$\lim_{z=\infty} z^n R = \lim_{z=\infty} \frac{z^n \cdot \theta_z M \cdot w^{n+1}}{(n+1)!} = 0.$$

Hence by definition

$$f(w) \sim f(0) + \frac{A_1}{z} + \frac{A_2}{z^2} + \cdots. \tag{5}$$

The form I have selected for the remainder in (1) readily applies for z complex. Taking the path of integration as a straight line from $z = 0$ to $z = w$ and writing $t = re^{i\theta}$, $w = w_0 e^{i\theta}$, then

$$|z^n| \cdot |R| \cdot \leqq \frac{|z^n| \cdot M}{n!} \int_0^{w_0} |w_0 - r|^n \cdot dr,$$

M being the maximum value of $|f^{(n+1)}(t)|$ in the interval of integration. But

$$\lim_{z=\infty} |z^n| \cdot |R| \leqq \lim_{z=\infty} \frac{|z^n| M \cdot {w_0}^n}{n!} \int_0^{w_0} dr = \lim_{z=\infty} \frac{z^n M \cdot {w_0}^{n+1}}{n!} = 0.$$

Thus (5) holds for z complex.

That the divergent series

$$f(0) + f'(0)z + \frac{f''(0)}{2!} z^2 + \cdots \tag{6}$$

is asymptotic at $z = 0$, or replacing z by $1/z$ in (6), at $z = \infty$, is evident since w may be chosen as $1/z$ in (5). (6) is therefore an asymptotic representation for $f(z)$ in the vicinity of $z = 0$, or if w be taken as $1/z$ in (5),

$$f\left(\frac{1}{z}\right) \sim f(0) + \frac{f'(0)}{z} + \frac{f''(0)}{2!\, z^2} + \cdots.$$

ANN ARBOR, MICH.,
January, 1913.

ON POINCARÉ'S CORRECTION TO BRUNS' THEOREM.

BY PROFESSOR W. D. MACMILLAN.

(Read before the American Mathematical Society, January 2, 1913.)

THE differential equations of motion for the problem of three bodies were first set up by Clairaut, and were published by him with the remark, "Let anyone integrate them who can." Clairaut himself had found ten of the eighteen integrals necessary for the complete solution of the equations, but in despair gave up the hope of finding any more, contenting himself with methods of approximation for those cases which were presented by our solar system, particularly, the motion of the moon. The solutions of these equations have engaged the attention of nearly all of the great mathematicians from Clairaut down to the present time, but no more integrals have been forthcoming. This universal failure has given rise, naturally, to a suspicion that there are no more integrals of a simple type, and this suspicion has been strengthened by the researches of Bruns and of Poincaré. In 1887 Bruns published his famous theorem* that the equations of motion of the problem of n bodies ($n > 2$) do not admit any integral which is algebraic in the rectangular coördinates and in the time, other than the ten classical integrals which were found by Clairaut. Bruns' theorem was soon followed by another† due to Poincaré. According to Poincaré's theorem the equations of motion of the problem of n bodies ($n > 2$) do not admit any uniform transcendental integral for values of the masses sufficiently small, other than the ten classical integrals. Comparing his own theorem with that of Bruns, Poincaré has said:‡ "The theorem which precedes is more general, in a sense, than that of M. Bruns, since I have shown not only that there does not exist any algebraic integral but that there does not exist even a uniform transcendental integral, and not only that an integral cannot

* *Acta Mathematica*, vol. 11 (1887).
† *Acta Mathematica*, vol. 13.
‡ Les Méthodes nouvelles de la Mécanique céleste, vol. 1, p. 253.

be uniform for all values of the variables but that it cannot remain uniform in a domain restricted as above. But, in another sense, the theorem of M. Bruns is more general than mine; I have established only, in effect, that there cannot exist algebraic integrals for sufficiently small values of the masses; and M. Bruns has shown that they do not exist for any system of values of the masses."

The demonstration of his theorem which was given by Bruns contained an error which was pointed out by Poincaré* in 1896, and the proper correction indicated by him. In order to see the nature of this correction it will be necessary to have an outline of the method by which Bruns achieved his demonstration.

Let the rectangular coördinates of the bodies be denoted by x_i, and let $dx_i/dt = y_i$. Bruns first observes that the differential equations

$$dx_i/dt = y_i, \quad dy_i/dt = f_i(x_i)$$

are algebraic in the variables x_i, y_i; and if a single irrationality s be introduced, the differential equations will be not only algebraic but also rational in the variables x_i, y_i, and s. The variable s is defined as a root of a certain algebraic equation

$$F(s;\ x_1,\ \cdots,\ x_n) = 0.$$

Considering first integrals which do not contain the time explicitly, it is shown that every integral must contain some of the variables y_i, and this is followed by the proof that the assumed algebraic integral can be built up of integrals which are rational functions of the variables x_i, y_i, and s, and consequently it is necessary to consider only integrals which are rational in these variables, e. g.,

$$\frac{G_1(x_i,\ y_i,\ s)}{G_2(x_i,\ y_i,\ s)} = \text{constant},$$

where G_1 and G_2 are polynomials in the arguments indicated and have certain homogeneity properties. It is then shown that the polynomials G_1 and G_2 satisfy the same differential equation

$$dG/dt = \omega G,$$

* *Comptes Rendus*, vol. 123, p. 1224.

where

$$\omega = \omega_1 y_1 + \omega_2 y_2 + \cdots + \omega_n y_n,$$

and the ω_i are rational in x_i and s, homogeneous of degree -1, and do not contain the variables y_i.

The polynomial G is now arranged according to powers of the y_i, thus

$$G = \psi_0 + \psi_2 + \cdots,$$

where ψ_0 contains the terms of highest degree in the y_i, ψ_2 is the ensemble of the terms of the next highest degree, etc. It is found that ψ_0, which is a homogeneous polynomial in the y_i, and also a homogeneous polynomial in the x_i and s, must satisfy the partial differential equation

$$\Sigma y_i \partial\psi_0/\partial x_i = (\omega_1 y_1 + \omega_2 y_2 + \cdots)\psi_0. \tag{1}$$

If ψ_0 does not contain the irrationality s, there exists a multiplier $m(x_i)$ which is a rational function of the x_i alone such that

$$m \cdot G = \varphi_0 + \varphi_2 + \cdots = \text{const.} \tag{2}$$

is an integral, where $\varphi_0 = m\psi_0$, etc., and φ_0 satisfies the partial differential equation

$$\Sigma y_i \frac{\partial \varphi_0}{\partial x_i} = 0. \tag{3}$$

If ψ_0 contains s, Bruns considered the product

$$\Psi = \prod_j \psi_0^{(j)},$$

where $\psi_0^{(j)}$ is the same as ψ_0 except that s is replaced by one of the other roots of $F(s, x_i) = 0$. Since the product Ψ is symmetrical in all the roots of $F = 0$, it is a rational function of the x_i. Consequently there exists a multiplier $H(x_i)$ such that $\Phi = H\Psi$ satisfies the equation

$$\Sigma y_i \partial\Phi/\partial x_i = 0. \tag{4}$$

From the character of the solutions of this equation, Bruns inferred that $\Sigma\omega_i dx_i$ of (1) was an exact differential even when ψ_0 contains s. It is necessary therefore only to consider integrals of the form (2).

Up to this point Bruns has used only the most general properties of the differential equations, viz., homogeneity and rationality, and from this point on the differential equations play a more important rôle. The next step consists in showing that if φ_2 [(equation (2)] is to be free from transcendental functions of the x_i, and to be a polynomial in the y_i, φ_0 must be a function of the ten classical integrals only. The assumed integral, mG = constant, is therefore compounded of the ten classical integrals plus another integral K which is of degree two less than mG in the y_i. The discussion of the integral K does not differ from that of mG. Its leading term must be built up from the ten classical integrals plus another integral K_1, and so on to the conclusion that mG is built up entirely of the ten classical integrals.

The case in which the assumed integral contains the time explicitly can be reduced to that in which the time does not occur explicitly.

The error committed by Bruns was in the character of the solutions of (4). The function Φ is a homogeneous polynomial in the y_i, and homogeneous in the x_i. Removing all factors from Φ which contain only the y_i and then taking $y_3 = y_4 = \cdots = 0$, Bruns arrived at a function Φ_{02} which satisfies the equation

$$y_1 \frac{\partial \Phi_{02}}{\partial x_1} + y_2 \frac{\partial \Phi_{02}}{\partial x_2} = 0.$$

The function Φ_{02} is homogeneous in y_1 and y_2. Bruns supposed it was also homogeneous in x_1 and x_2, while as a matter of fact it is homogeneous in $x_1, \cdots, x_n$. The conclusion drawn by Bruns that $\Sigma \omega_i dx_i$ must be an exact differential is not correct, and Poincaré gave an example in which it is not verified. But Poincaré remedied this defect by showing that while in general there exist functions ϕ_0 which satisfy the conditions imposed upon it and which satisfy equation (1), without satisfying the condition $\Sigma \omega_i dx_i =$ an exact differential, such functions cannot arise from the astronomical problem. Poincaré did not give the details of his analysis and sketched his proof only in its broadest outlines. The details of this proof have been given by Forsyth,* but the proof given by Forsyth is open to the objection that while the y_i are constants so far as the x_i are concerned in the partial differential equation

* Theory of Differential Equations, vol. 3, p. 351 et seq.

(1), they have not been consistently regarded as such by Forsyth. An excellent exposition of this proof has been given by Whittaker,* but while Whittaker has avoided the errors of Forsyth he has committed one of his own.

The function $\Phi = \varphi_0(s_1) \cdot \varphi_0(s_2) \cdot \varphi_0(s_3) \cdots$ is a rational homogeneous function of the x_i and a homogeneous polynomial in the y_i. The factors $\varphi_0(s_j)$ differ from one another only in the roots s_j; consequently, two factors become equal when two of the s_j become equal. Suppose $\varphi_0(s_1) = \varphi_0(s_2)$; then $s_1 = s_2$ defines a relation between the x_i, say

$$f(x_1, \cdots, x_n) = 0. \tag{5}$$

For values of the x_i lying on $f = 0$, the factor $\varphi(s_1) = \varphi(s_2)$ and consequently Φ has a double factor. Let us think of $y_2, \cdots, y_n$ as fixed, or given arbitrarily; then y_1 can be determined so as to satisfy the two equations

$$\Phi = 0, \quad \partial\Phi/\partial y_1 = 0, \tag{6}$$

and consequently also† $\partial\Phi/\partial x_1 = 0$. In fact, the partial derivative with respect to any of the variables will vanish if equations (6) are satisfied.

Since (5) is a condition of equal roots of $\Phi = 0$ it follows that $f(x_1, \cdots, x_n) = 0$ is the eliminant of (6), or a factor of the eliminant. Consequently, by the theory of elimination, there exist multipliers, A and B, such that

$$f \equiv A\Phi + B\,\partial\Phi/\partial y_1. \tag{7}$$

On differentiating (7) with respect to x_i it is found that

$$\frac{\partial f}{\partial x_i} \equiv A\frac{\partial \Phi}{\partial x_i} + B\frac{\partial^2 \Phi}{\partial y_1 \partial x_i} + \Phi\frac{\partial A}{\partial x_i} + \frac{\partial \Phi}{\partial y_1}\cdot\frac{\partial B}{\partial x_i}.$$

Multiplying through by y_i and adding with respect to i, there results

$$\sum_i y_i \frac{\partial f}{\partial x_i} \equiv A\sum y_i \frac{\partial \Phi}{\partial x_i} + B\sum y_i \frac{\partial^2 \Phi}{\partial y_1 \partial x_i} + \Phi\sum y_i \frac{\partial A}{\partial x_i} + \frac{\partial \Phi}{\partial y_1}\sum y_i \frac{\partial B}{\partial x_i}. \tag{8}$$

* Analytical Dynamics.

† This relation seems to have been overlooked by Whittaker.

Consider now the various terms of the right member of (8): From (4) it is seen that $\Sigma y_i \partial\Phi/\partial x_i \equiv 0$; and from (6), $\Phi = 0$ and $\partial\Phi/\partial y_1 = 0$. On differentiating the identity $\Sigma y_i \partial\Phi/\partial x_i \equiv 0$ with respect to y_1 it is seen that

$$\frac{\partial\Phi}{\partial x_1} + \Sigma y_i \frac{\partial^2\Phi}{\partial y_1 \partial x_i} \equiv 0$$

and since $\partial\Phi/\partial x_1 = 0$ it follows that $\Sigma y_i \partial^2\Phi/\partial y_1 \partial x_i = 0$. Hence the right member of (8) vanishes and we have the result that values of the x_i and y_i which satisfy (5) and (6) also satisfy the equation

$$\Sigma y_i \frac{\partial f}{\partial x_i} = 0. \tag{9}$$

Since Φ satisfies the equation $\Sigma y_i \partial\Phi/\partial x_i \equiv 0$ it involves the x_i only through the expressions $x_i y_1 - x_1 y_i$, $(i = 2, \cdots, n)$. Let us define new variables v_i by the relations

$$v_i = x_i + y_i \tau \qquad (i = 1, \cdots, n). \tag{10}$$

The variables v_i can be regarded as the coördinates of a straight line in space of $3n$ dimensions, the line passing through the point whose coördinates are the x_i, the slopes of the line being defined by the y_i. One sees that

$$v_i y_1 - v_1 y_i = x_i y_1 - x_1 y_i,$$

and hence if the point x_i satisfies $\Phi = 0$, so also do all the points v_i as defined by (10); and this is the justification of Poincaré's remark that $\Phi = 0$ represents an aggregate of straight lines in space of $3n$ dimensions.

Consider now any line $v_i = x_i + y_i \tau$, where the x_i lie on the surface $f = 0$, and the y_i are such as to satisfy (6). Its intersections with the surface $f = 0$ are given by the equations

$$f(x_i + y_i\tau) = 0 = f(x_i) + \frac{\partial f}{\partial \tau} \cdot \tau + (\cdots)\tau^2, \tag{11}$$

or

$$0 = f(x_i) + \tau\Sigma \frac{\partial f_i}{\partial x_i} y_i + \tau^2(\cdots);$$

but since $f(x_i) = 0$ and $\Sigma y_i(\partial f/\partial x_i) = 0$ it is seen that $\tau = 0$ is a

double root and consequently the line is tangent to the surface $f = 0$. Thus all the lines which belong to the double factor of Φ are tangent to the surface $f = 0$.

Now let the x_i and y_i have such values that they satisfy $\Phi = 0$ but the point x_i does not necessarily lie on $f = 0$, and consider the totality of lines $v_i = x_i + y_i\tau$ which are tangent to $f = 0$. They are given by the equations

$$f(x_i + y_i\tau) = 0, \quad \frac{\partial}{\partial\tau} f(x_i + y_i\tau) = 0. \tag{12}$$

The τ-eliminant of equations (12) represents the totality of lines tangent to $f = 0$. Hence it includes the two or more factors of Φ which become equal when the x_i satisfy $f = 0$. Since the eliminant is rational and Φ is irreducible, the eliminant must be Φ itself or a multiple of it.

In the astronomical problem the equation $F = 0$ which defines the roots s_j is known. The surfaces $f = 0$ are therefore readily determined and all possible functions Φ can be found. To satisfy the conditions which Bruns has stated, Φ must be factorable into real factors which are polynomials in the y_i and rational in the x_i and s. It is found upon examination that there does not exist a Φ which satisfies all these conditions and consequently the original φ_0 with which we set out cannot contain s. Therefore Bruns' conclusion that we need consider only integrals of the type (2) was correct, even though his argument was wrong. The integrity of the theorem has been preserved by the penetrating insight of Poincaré.

UNIVERSITY OF CHICAGO,
January 8, 1913.

NOTE ON THE GROUPS FOR TRIPLE-SYSTEMS.

BY MISS L. D. CUMMINGS.

THE method of "Triple-systems as transformations and their paths among triads," given by Professor White in the *Transactions*, volume 14 (1913), page 6, has been applied by me to the two following triple-systems on fifteen elements. The results obtained agree with the fact, which I had discovered previously by a different method of analysis, that two non-congruent triple-systems may have the same group.

I.

1	2	3	1	8	9	2	8	12	3	8	15	4	7	15	5	9	10	7	8	14
1	4	14	1	10	15	2	9	15	3	9	12	4	9	11	5	12	14	7	9	13
1	5	13	2	4	13	2	11	14	3	10	14	4	10	12	6	9	14	8	10	13
1	6	11	2	5	7	3	4	5	3	11	13	5	6	15	6	12	13	11	12	15
1	7	12	2	6	10	3	6	7	4	6	8	5	8	11	7	10	11	13	14	15

II.

1	2	5	1	9	13	2	7	14	3	6	15	4	8	10	5	13	14	8	9	14
1	3	8	1	11	12	2	8	13	3	11	13	4	11	14	6	7	9	9	10	11
1	4	15	2	3	9	2	11	15	3	12	14	5	6	11	6	8	12	9	12	15
1	6	14	2	4	12	3	4	7	4	5	9	5	7	12	7	8	11	10	12	13
1	7	10	2	6	10	3	5	10	4	6	13	5	8	15	7	13	15	10	14	15

The group for each of the above triple-systems is of order 3 and is generated by the substitution

$$s = (1\ 7\ 14)\ (2\ 6\ 10)\ (3)\ (4\ 12\ 8)\ (5\ 9\ 15)\ (11)\ (13).$$

February, 1913.

DE SÉGUIER'S THEORY OF SUBSTITUTION GROUPS.

Théorie des Groupes finis. Eléments de la Théorie des Groupes de Substitutions. Par J. A. DE SÉGUIER. Paris, Gauthier-Villars, 1912. x+228 pp. 10 Fr.

THE earliest separate book on the theory of groups is Jordan's Traité des Substitutions, which appeared in 1870 and is still one of the most valuable treatises along certain lines. Twelve years later there appeared Netto's Substitutionentheorie, which was translated into Italian by G. Battaglini, in 1885, and into English by F. N. Cole, in 1892. Five years after this English translation, W. Burnside published the first separate treatise on groups originally written in our language, under the title Theory of Groups of Finite Order. A second and enlarged edition of this work appeared in 1911. In 1900 the first printed edition of a book on this subject originally written in Italian appeared under the title Lezioni sulla Teoria dei Gruppi di Sostituzioni, by L. Bianchi.

Since the beginning of the present century new books on the theory of groups of finite order have appeared more rapidly, as may be seen from the following list: L. E. Dickson, Linear Groups, 1901; J. A. de Séguier, Groupes abstraits, 1904; H. Hilton, Introduction to the Theory of Groups of Finite Order, 1908; E. Netto, Gruppen und Substitutionentheorie, 1908;

J. A. de Séguier, Eléments de la Théorie des Groupes de Substitutions, 1912. Among the other works which have appeared during the same period, and which tend to make this theory more easily accessible, we may mention the extensive articles in the Encyclopédie des Sciences mathématiques and in Pascal's Repertorium, and the bibliographies published by B. S. Easton in 1902 and by C. Alasia in 1908–1909.

The book under review is largely based on the Groupes abstraits by the same author, which was reviewed by L. E. Dickson in this BULLETIN, volume 11 (1904–1905), pages 159–162. The closing lines of this review are as follows: "The reader of the present volume will be impressed with the author's complete mastery of his subject and will find in it a useful compact summary of the results to date in the purely abstract part of finite group theory." A similar general verdict applies to the volume before us. Although this volume is somewhat larger than its predecessor, it is written in the same compact style and contains a very large amount of information. Many readers would doubtless prefer a less compact form and a less extensive use of special symbols; but after these symbols have been mastered, they tend towards clearness as well as towards brevity.

Just as in the Groupes abstraits the number of minor errors in the present volume is large. Lists of such errors together with some complements relating to both of these volumes appear near the end of the present volume. These lists under the heading "Additions et Corrections" cover eleven closely printed pages. It is evident that such a multitude of minor errors must be a source of much confusion to those who are not fully familiar with the subjects treated. In fact, the reviewer feels that both of these volumes are better suited for the student who has already arrived at considerable maturity in the theory of groups than for the beginner. The tendencies to begin with the general instead of the special cases and to express the results very concisely have many advantages but they present too many difficulties for most beginners.

As a work of reference the present volume offers some advantages over its predecessor since it contains an "Index des terms" and an "Index des notations." Together these two indices cover only two pages, and there is no subject index. In view of the fact that many of the most modern develop-

ments are considered, an author index would also have rendered useful service in several ways, There are no lists of exercises, but numerous illustrative examples are given, and many references, especially to the Groupes abstraits, are found in the text.

The volume consists of five chapters bearing the following headings in order: Substitutions; Groupes de substitutions, théorèmes généraux; Representation des groupes par des groupes de substitutions; Groupes de degré n et de classe $n-1$, groupes linéaires; and Groupes de degré kp, $p+\alpha$, $2p+\alpha$. These five chapters are followed by three "notes" covering 45 pages, and bearing the following headings: Sur la théorie des matrices, Equations algébriques, and Sur les groupes de degré n et de classe $n-1$. The second of these "notes" is the longest, and it treats, in a concise form, the general Galois theory of algebraic equations.

The concept of substitution is based upon a (1, 1) correspondence between the elements of a set (ensemble), and a substitution is said to consist of the operation of replacing an element by the corresponding element in such an automorphism. A substitution is said to be normal if the cycles are separated by periods. For instance, the substitution $(abcf)(dgh)(ke)$ may be represented in a normal form as follows: $abcf \cdot dgk \cdot ke$. This form has been employed by several other writers and it appears to the reviewer as the most convenient notation for substitutions in the restricted sense.

The fact that the technical terms are not always defined before they are employed in the present volume may be due to a desire to avoid, as far as possible, the repetition of definitions given in the Groupes abstraits. As instances of this fact, we may state that the technical terms normale and isomorphes are used on page 4, but these terms have not been defined on the preceding pages. The latter part of the first chapter is devoted to a consideration of polynomials representing substitutions in a Galois field, and it naturally contains a number of results due to L. E. Dickson.

The second chapter begins with a consideration of some of the fundamental properties of imprimitive and of intransitive groups. It is observed that the substitutions of each transitive constituent of an intransitive group must constitute a group, and these constituents are then combined into two sets denoted by A and B respectively. It is asserted, on page 28, that

each of the factor groups of the entire group is a factor group of A or of B. This statement is evidently incorrect since the entire group may have factor groups whose orders exceed the order of each of the constituent groups A and B. The chapter closes with a consideration of relations between the class, degree, and order of a group.

The proposition that every abstract group can be represented in an infinite number of different ways as a substitution group constitutes the opening sentence of Chapter III. Instead of pursuing the usual method of first proving that every finite group can be represented as a regular substitution group, our author starts with the general proposition and establishes it directly. An abstract group is said to be represented properly by a given substitution group if there is a (1, 1) correspondence between these groups; and the representation is said to be of the first, second, or third species as the augmented co-sets are multiplied on the right or left, or are transformed by all the operators of the abstract group, to obtain the substitutions of the representing group. Different simple isomorphisms between an abstract and a substitution group which represents it properly are said to be different representations of this group.

A footnote on page 86 is devoted to a consideration of an erroneous theorem announced by A. L. Cauchy as regards simply transitive primitive substitution groups. According to this theorem a primitive group of degree n cannot be simply transitive unless $n-1$ is divisible by the degree of each transitive constituent of the subgroup composed of all the substitutions which omit a given letter. Hence a primitive group of degree $p+1$, p being an odd prime, could not be simply transitive. The author of the present work calls attention to the facts that the first part of this theorem is easily seen to be incorrect and that the latter part has been verified for $p \leqq 13$ by C. Jordan and others, but he does not state that for $p = 83$ it has been proved that this part is also incorrect. Cf. G. A. Miller, *Bibliotheca Mathematica*, series 3, volume 10, 1910, page 321.

Chapter IV begins with a consideration of the transitive substitution groups of degree n in which all the substitutions except identity involve at least $n-1$ letters. The greater part of the chapter is, however, devoted to a study of linear groups. A number of abstract definitions of well-

known groups are incidentally developed, and simple isomorphisms between various groups are investigated. The chapter closes with a determination of the groups which can be represented on a prime number p of letters and which involve exactly $p+1$ subgroups of order p.

The last chapter starts with a proof of the important theorem due to Burnside, which states that a transitive group on p letters must be multiply transitive whenever it involves more than one subgroup of order p. The proof is based upon the one given by I. Schur in the *Jahresbericht der Deutschen Mathematiker-Vereinigung*, volume 17 (1908), page 171. This is followed by a study of the interesting theorems relating to the multiply transitive groups of degree $p+\alpha$ which involve subgroups of order p. The latter part of the chapter is devoted to a study of the well known multiply transitive groups due to Mathieu.

In the preface we are told that the present volume is devoted to the substitutions which may be called natural, that is, to the substitutions on a finite number of objects whose order is simple. The author enters only partly into the field of linear modular groups. A more profound study of these groups, and a determination of systems of solvable groups, constitute the subjects of a proposed later volume by the same author. It is to be hoped that this later volume will contain at least a subject index, including the material of the present volume and of the earlier volume on Groupes abstraits. Such an index would make these volumes much more valuable for reference. A general author index would also render useful service.

G. A. MILLER.

WILSON'S ADVANCED CALCULUS.

Advanced Calculus: A Text upon Select Parts of Differential Calculus, Differential Equations, Integral Calculus, Theory of Functions, with Numerous Exercises. By EDWIN BIDWELL WILSON. Boston, Ginn and Company, 1912. ix+566 pp.

SOME years ago Professor Asaph Hall, after reading carefully Poincaré's Mécanique Céleste, which had just been published, wrote to its distinguished author and took him severely to task because he had devoted his splendid mathematical knowledge

and ability not to the advancement of astronomical science or to the invention of improved methods of investigation, but to mere criticism.

One of my friends, a brilliant physicist and a learned and skilful mathematician, has frequently said concerning the "exact mathematics" of our day that what we call "rigor" will be regarded by our descendants as we regard the rigor of our ancestors, an opinion by the way very strongly hinted at by Professor Bôcher in his St. Louis address.

We live in a critical age. Our present mathematical ideal is to draw from a carefully framed set of definitions and postulates by what we consider rigorously logical reasoning conclusions which seem to us necessary.

In addition to critical mathematicians, pure logicians to whom intuitive methods are like a red rag to a bull, there have always been creative mathematicians, often careless fellows who had a surprising knack for getting correct results by methods quite open to criticism; and mathematical artisans,—astronomers, physicists, engineers—to whom mathematics is a tool, and who are much more interested in results than in methods.

Far be it from me to disparage the critic; he is a useful person even if he is apt to be a little intolerant; but let us not train all our promising young men as if criticism were the only thing or even the main thing to work toward. Perhaps some of them may develop creative power; certainly many of them will become mathematical artisans; and life is short.

In our teaching let us by no means disregard the modern methods and the modern spirit. Let us familiarize the student with the present notions of rigor and with the received critical logical methods, but let us not stop there. Let us train him to use intuitive methods freely and to cultivate his powers of invention, and in case of need to work rapidly and even recklessly. If properly taught he will be in position to check his results and processes if they are challenged or if he doubts their accuracy, and yet he will have gained the power and confidence that will carry him forward swiftly and in the main safely. Better an occasional mistake through over-confidence than a perfectly safe snail's pace over ground every foot of which he has carefully tested before he has dared trust it with his weight.

That this is the ideal toward which the author of the book

under review has striven is clear from his preface. "It has been fully recognized that for the student of mathematics the work on advanced calculus falls in a period of transition—of adolescence—in which he must grow from close reliance upon his book to a large reliance upon himself. Moreover, as a course in advanced calculus is the ultima Thule of the mathematical voyages of most students of physics and engineering, it is appropriate that the text placed in the hands of those who seek that goal should by its method cultivate in them the attitude of courageous explorers, and in its extent supply not only their immediate needs, but much that may be useful for later reference and independent study.

With the large necessities of the physicist and the growing requirements of the engineer, it is inevitable that the great majority of our students of calculus should need to use their mathematics rapidly and vigorously rather than with hesitation and rigor. Hence, although due attention has been paid to modern questions of rigor, the chief desire has been to confirm and to extend the student's working knowledge of those great algorisms of mathematics which are naturally associated with the calculus." In my opinion the ideal is an admirable one; it seems to me that the author's approximation to that ideal has been a remarkably close one.

The first chapter, Review of fundamental rules, contains a masterly sketch of the groundwork of the calculus, differential and integral, with which the student is supposed to be familiar when he begins the study of the book, and serves as a guide and as an aid to him in the careful review of his elementary course which perforce he must make before going on to the higher parts of his subject. In this chapter the proofs are often merely hinted at, or if given are put in the briefest possible form without any attempt to avoid the use of intuition or to strive at modern rigor.

The second chapter, Review of fundamental theory, is a chapter in "exact mathematics." In the words of the author "the object of the chapter is to set forth systematically, with attention to precision of statement and accuracy of proof, those fundamental definitions and theorems which lie at the basis of calculus, and which have been given in the previous chapter from an intuitive rather than a critical point of view."

It is needless to say that for most students beginning advanced calculus this chapter is in no sense a review. It is new

work and hard work. Such matters as the Concept and theory of real number (very briefly set forth); Definition of a limit; Theorems on limits and on sets of points; Real functions of a real variable; Continuity; Uniform continuity; Differentiability; Rolle's theorem and the theorem of the mean; Summation and integration; Integrability, are taken up from the modern point of view and the modern rigorous theories and proofs are carefully and well given.

During the rest of the book references are freely made to this chapter, and occasionally an important or fundamental proof is put into modern rigorous form, but in the main there is a refreshing absence of epsilons and deltas and the rest of the paraphernalia of the critical mathematician.

As an avowed treatise on advanced calculus the book begins with Chapter III, and is almost encyclopædic in its range. Topics treated exhaustively, topics briefly sketched, topics merely hinted at and illustrated or suggested by problems chosen from the fields of pure analysis, of mechanics, of engineering, and of physics are almost without number, and are by no means fully revealed by the excellent table of contents, or even by the uncommonly detailed index.

To the teacher or to the working mathematician the work is invaluable. It probably was not written for the unaided student. He would certainly find it too condensed and too difficult. In the hands of a skilful teacher it might be an effective text book, but even then the class would probably find it rather hard sledding.

The labor of preparing the book must have been enormous and the author deserves the thanks of the mathematical public for a most valuable addition to the literature of the calculus.

W. E. Byerly.

THE CALCULUS IN INDIA.

A Text-book of Differential Calculus. By G. Prasad. Longmans, Green and Co., 1909. xii+161 pp.

A Text-book of Integral Calculus. By G. Prasad. Longmans, Green and Co., 1910. x+241 pp.

Twin texts on calculus from Benares, Holy City of the Hindus! If introduced in this country they would be pro-

ductive, we fear, of very unholy comments by the students; for they are hard and mathematical, and pay but small respects to the feeble intellect that longs for a nurse and for practical applications, in short for being entertained, from the start. Yet these texts have most excellent characteristics, chief among which from our point of view is their difference from the ordinary run of texts. The volumes are "intended for beginners and so designed as to meet the requirements of Part I of the Cambridge Mathematical Tripos Examinations, and of the Examinations for the B.A. and B.Sc. degrees of Indian Universities."

The author has gone to the limit in abolishing the infinitesimal and differential from both differential and integral calculus. At first we thought that there was no mention at all of infinitesimals, but finally we found, the last thing in the Differential Calculus, in fine type, as a footnote to the chapter on indeterminate forms, the definitions of infinitesimal and infinites, with a statement that they were not of much importance to the beginner. So far the definition of differential has proved elusive, though the symbol is used in d/dx and in $\int dx$. The idea of a limit of an infinite sum of infinitesimals is relegated to a short chapter by itself, starred as difficult, and is not used in the applications; alas, poor Duhamel. But why not? In many recent texts on calculus the infinitesimal and differential are, for the sake of precision, bereft of all their fecundity and of all their effectiveness and naturalness as a method of thought for the student. When thus slighted it is as well to discard them utterly.

The two volumes together, apart from appendices and answers, total 345 pages. The pages are wide open, plenty of unused paper, scarcely a useless word. The text is therefore really short. And this is the more noteworthy in view of the amount of space the author will give to rubbing in some fundamental process. Thus in most books when $\sin x$ has been differentiated, the formulas for differentiating other trigonometric functions and the inverse functions are derived by various devices. Not so with Prasad; he takes each one and determines ab initio the limit of $[f(x+h)-f(x)]/h$. He then gives exercises where the student may do likewise, and it is not until the following chapter that he develops the rules for sums, products, quotients, functions of functions, and the like. There are numerous and varied exercises upon which the student may practice formal differentiation.

Chapter IV deals with tangents and normals in rectangular and polar coördinates, and such allied questions as subnormals. The author derives the equation of the tangent and appends two notes, in the first of which he remarks that the derivative is the slope of the tangent, and in the second derives the expression for ds/dx. The method is solely one of limits; personally we have a preference for infinitesimal figures which show such formulas at a glance and afford a visual means of remembering them. A chapter on asymptotes follows, and then one on curvature. The center of curvature is defined as the limiting position of the intersection of a fixed normal and a variable normal approaching the fixed one as a limit. Thus the center of curvature is found first, the radius subsequently. The treatment extends to polar coördinates, to involutes and evolutes, and to concavity, convexity, and points of inflection. There is a brief chapter on envelopes and a long one on curve tracing and on properties of special curves.

Although the symbol for the second derivative has been defined and used in connection with curvature, the author comes only now to his chapter on successive differentiation, and he makes it a real chapter, not a mere note with silly exercises. He finds the nth derivative of x^m, a^x, $\sin x$, $\cos x$, $e^{ax} \sin bx$, $e^{ax} \cos bx$, of a rational fraction (by decomposition into partial fractions), of products (Leibniz's formula), and of various functions for which a recurrence formula may be established. The good student will learn a lot out of this chapter, the poor student, we fear, will be unable to do anything—although the author states that these texts are based upon his experience in teaching a large number of pupils. We are now ready for Rolle's theorem and the finite or infinite developments of functions by Taylor's or Maclaurin's theorems; and the careful work on successive differentiation enables us to expand a great many functions—$e^{a \sin^{-1} x}$, for example—other than the common elementary functions. The volume closes with a brief account of maxima and minima, and indeterminate forms. There are short notes on Weierstrass's continuous non-differentiable function, on Rolle's and Taylor's theorems, and on partial differentiation.

The Integral Calculus begins with the definition of the indefinite integral, and of the definite integral as the difference of the indefinite between limits—a dangerous definition, as it

makes the integral of $1/x^2$ from -1 to $+1$ equal to -2. It is stated that the definite integral is a limit of a sum, and the reader is left to verify the fact by calculating the limit for some simple functions and comparing it with the value obtained by integration. Then follows a long chapter on fundamental methods of integration—integration by substitution, by parts, and by reduction formulas. The author does not use the differential method as

$$\int \frac{dx}{x \log x} = \int \frac{d \log x}{\log x} = \log \log x,$$

but the method of substitution as $x = \varphi(t) = e^t$, $\varphi'(t) = e^t$,

$$\int f(x)dx = \int f(x) \frac{dx}{dt} dt = \int \frac{1}{t} dt = \log t = \log \log x.$$

Moreover, he uses the formula for integration by parts as

$$\int uvdx = uV - \int u'Vdx,$$

where V is the integral of v.

For those who are never to separate the derivative into its differentials and those who make the separation only at that late stage when the student is beginning integration, and has enough difficulties with integration alone without having a new notation for differentiation, this method is to be recommended. Indeed the author might more logically have used D for differentiation (and D_t when the differentiation was performed with respect to some variable t other than the apparent one) and $\int$ without the dx for integration ($\int_t$ when the variable is t). We believe, however, that the differential method is better, and hope it has suffered only a temporary total or partial eclipse. We believe that Huntington, in his syllabus of calculus for the Society for the Promotion of Engineering Education, has cast a line to the process of differentiation, as contrasted with derivation; we trust he may rescue and resuscitate it.

Prasad next gives a systematic treatment to the integration of algebraic rational and irrational fractions. This is starred as more difficult and possible of omission in the first reading. Then follows a chapter on integrating transcendental functions. In the next chapter we come to definite

integrals, their definition as the limit of a sum, their general properties, and their evaluation in a few simple cases. The work thus far covers a little more than half the volume; the remaining portion is given to various applications.

The remarkable thing about all these applications is the complete omission of any ideas concerning limits of sums. The method is always to find a derivative, and then integrate. For example, the result for ds/dx has been found; hence s may be obtained. In like manner $dA/dx = y$ may be established, and hence A is the integral of y. And so on, to arcs and areas in polar coördinates, to surfaces and volumes of revolution, to centers of gravity, centers of pressure, moments of inertia, and attractions. All are treated by differentiation. Why not? Why not eliminate the troubles connected with limits of sums? The author has made the presentation clear and rigorous, and has shown conclusively that we do not need to bother with the integral as a limit of a sum in elementary calculus. His method is worthy of our most serious consideration—if we desire to be rigorous instead of suggestive, and we can hardly be both in a first course on calculus.

The remaining applications are to the dynamics of a particle, prefaced by a few sections on the integration of the simpler differential equations. There are notes on the integration of infinite series, on Riemann's discontinuous integrable function, and on Fourier's series.

These texts merit our special consideration because they are different from those we are used to. It would be interesting to see them tried on American classes both for the effect on the students and for the effect on the teachers.

EDWIN BIDWELL WILSON.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY.

SHORTER NOTICES.

Mémoires Scientifiques. By PAUL TANNERY. Publiés par J. L. HEIBERG and H. G. ZEUTHEN. I. *Sciences Exactes dans l'Antiquité*, 1876–1884. Toulouse, Edouard Privat; Paris, Gauthier-Villars, 1912. xix+465 pp. Price 15 francs.

IN our time there have been three men whose love for ancient science and whose perfect command of the Greek language

fitted them above all their contemporaries to write upon the subject of the mathematics of Greece. These men are Paul Tannery, Sir Thomas Heath, and Professor Heiberg. If to this trio another should be added, it might well be Professor Zeuthen. It is, therefore, very proper that the scientific memoirs of Paul Tannery should be edited by Professors Heiberg and Zeuthen, for no others are better fitted to undertake the task con amore, nor could any scholars be found more adequately prepared for the labor.

Paul Tannery was a rare genius, one to whom Greek was a second language, and one who lived his intellectual life in the contemplation of the science of antiquity. Never properly recognized by his country, employed in the government manufacture of tobacco instead of being offered a chair in the university, he nevertheless produced a series of memoirs that would have brought honor to any higher institution of learning and that will give him an enduring place among the scholars of France. He was a prominent member of the third International Congress of Mathematicians at Heidelberg, in 1904, and seemed at that time to be in his prime; but it was only a few weeks later that the news of his sudden death appeared. He passed away too soon for his greatest scientific success, and his loss was a sad one for the scientific study of the history of mathematics in France.

When it was planned to bring together his memoirs, the talented Madame Tannery secured the coöperation of Professors Heiberg and Zeuthen in editing the work, and the promise of his brother Jules Tannery to write an introduction to the first volume. The premature death of the latter, in November, 1910, deprived the Ecole normale supérieure of its leader, and at the same time prevented the carrying out of this plan. Thus within the short space of six years these two brothers passed away, each in the prime of life, and each seeming to leave a great work unfinished.

Paul Tannery began his contributions in 1876, and the present volume contains such memoirs as appeared on the exact sciences of antiquity from that date until 1884. The next two volumes will continue this topic, bringing the memoirs down to the time of his death. The subsequent volumes will relate to the exact sciences among the Byzantines (volume IV), the exact sciences in the middle ages (volume V), pure mathematics (volume VI), philosophy (volume VII), classical philology

(volume VIII), and reviews (volume IX). There will also be a volume (volume X of the series) containing his biography, a bibliography, and selections from his correspondence.

In editing the memoirs advantage has been taken of manuscript notes on the margins of the author's copies, and these have been marked in such a way as to show that they did not form part of the original publications. The editors' notes have also been marked in a special manner so that the responsibility for them is easily placed. In general, however, the articles and footnotes stand as they were written, a plan of editing which every student of the history of mathematics will commend.

Tannery's memoirs form a basis for a history of Greek mathematics, although in themselves they do not form a connected narrative. The mind of the author did not work along the lines of continued narration; he preferred to attack isolated problems and solve them. His is the material out of which lesser writers make their histories, and no one has done more than he to furnish material on Greek mathematics for the use of some future Montucla.

The memoirs in the first volume are twenty-nine in number. They include a wide range of subjects, beginning with the astronomical system of Eudoxus and ending with the " modius castrensis,"—the Italicus modius or Italicus sextarius mentioned in Diocletian's edict De pretiis rerum venalium. Some of the more interesting and important topics considered are the nuptial number of Plato, which Tannery takes to be 2,700; the lunes of Hippocrates; the solution of the Delian problem by Archytas and Eudoxus; the question of the date of Diophantus, based in part upon a study of the varying cost of the wine mentioned in the problems of the Arithmetica; the Greek arithmetic as set forth by Pappus and Heron; the Eudemian fragments; the origin of the proof of nines, which he assigns to the Greeks instead of the Orientals; the method of Archimedes in measuring the circle; the solution of the quadratic before Euclid; and the stereometry of Heron.

It is unnecessary to speak further of the contents of these memoirs, since the latest of them was published thirty years ago and all are therefore known to historians of mathematics. It may suffice to say that the volume is a credit to the editors and publishers as well as to the distinguished scholar whose articles it contains.

DAVID EUGENE SMITH.

Die logischen Grundlagen der exakten Wissenschaften. By PAUL NATORP. (Wissenschaft und Hypothese, XII.) Leipzig and Berlin, B. G. Teubner, 1910. xx+416 pp.

Probleme der Wissenschaft. Part I: *Wirklichkeit und Logik.* Part II: *Die Grundbegriffe der Wissenschaft.* By FEDERIGO ENRIQUES. Translation by KURT GRELLING. (Wissenschaft und Hypothese, XI.) Leipzig and Berlin, B. G. Teubner, 1910. x+vi+599 pp.

Erkenntnistheoretische Grundzüge der Naturwissenschaften. By PAUL VOLKMANN. (Wissenschaft und Hypothese, IX.) Second Edition. Leipzig and Berlin, B. G. Teubner, 1910. xxiii+454 pp.

A PHILOSOPHER devotes the greater part of his book to the foundations of mathematics; a mathematician boldly attacks the problems of philosophy; a physicist lectures on the theory of knowledge. These phenomena, as well as the fact that the books in question are sent to a mathematical journal for review, may well be regarded as significant of our times. Recent progress in the foundations of mathematics and the revolutionary conceptions and theories of present-day physics have necessarily struck deep into the current of philosophical thought. The authorship of these books need then occasion no surprise.

An extended notice of books of this character in a mathematical journal seems hardly called for as yet, however, as they will probably be of interest only to a limited number of mathematicians. A brief indication of the nature of their contents, however, may be given.

Professor Natorp devotes the first 97 pages of his book to an exposition of what he regards as the fundamental problems of logic. He follows Kant in insisting that the primitive act of thought is synthetic and repudiates vigorously the attempt of some logicians to base logic on a meaningless symbolism. In fact, logic as such is not and can not be a deductive science at all. This point of view leads necessarily to a genetic theory of knowledge in which the process or method of thought is the determining factor of knowledge.

The author regards as a primitive faculty of the mind the power of conceiving any mental act to be repeated indefinitely. He thus obtains essentially what mathematicians would call the abstract form of an unlimited sequence. On this he

bases his development of the notion of integral numbers, and thence by changing his " unit " the notion of rational numbers, together with that of the fundamental operations on numbers (pages 98–159). Then follows (pages 160–224) a very careful and illuminating discussion of the problems involved in the notions of irrational numbers and continuity. The ideas of Dedekind, Weierstrass, Cantor, Pasch, and Veronese are considered in much detail. In the next chapter (pages 225–265) the concepts of direction and dimensionality are developed as attributes of the notion of number, which is followed (pages 266–325) by a discussion of time and space as mathematical concepts. The foundations of geometry receive attention here. In the last chapter (pages 326–404) the author deals with the time-space order of natural phenomena and the mathematical foundations of science. Here we find a discussion of the fundamental concepts and laws of mechanics, the principle of the conservation of energy, etc. The book closes with a discussion of the principle of relativity, in which the author finds a final justification of an idealistic philosophy.

Professor Enriques' two volumes will be examined with much pleasurable anticipation by any mathematician interested in philosophical questions. One of the most interesting features of the books is the fact that the author applies mathematical terminology to the formulation and discussion of his problems—he speaks a language, therefore, which is intelligible to a mathematician. His program is an ambitious one, nothing less in fact than to seek the common elements in the various branches of scientific activity, to seek a general point of view from which all science, in its broadest sense, may be unified with reference both to the formulation of its problems and the development of its methods. The reader cannot fail to be struck with the wide range from which the author has drawn his illustrations. The chapter headings will perhaps serve to give an idea of the general plan of the work: I. Introduction; II. Facts and theories; III. The problems of logic (pure logic, the applications of logic, the physiological aspects of logic); IV. Geometry (significance of geometry, psychological genesis of geometrical concepts); V. Mechanics; VI. Extension of mechanics (physics as an extension of mechanics, the mechanistic hypothesis and the phenomena of life). The first volume contains the first three chapters; the second, the last three. We may add that the translation from the Italian original seems exceptionally well done.

The first edition of Professor Volkmann's book was reviewed in this BULLETIN, volume 4 (1898), page 355. What was said there will apply very well to the new edition. It seems superfluous, then, to do more than indicate very briefly the changes that have been made in the new edition. What in the first edition was the first lecture has in the second been incorporated in the closing lectures. The new edition begins with a historical retrospect on the development of science and its conceptions. A section on the subjective and objective elements in knowledge has been added, as well as an appendix containing two earlier papers of the author giving an appreciation, from the philosophical point of view, of the Newtonian system of mechanics. The book closes with a complete bibliography of the author's philosophical writings and very complete author and subject indexes.

J. W. YOUNG.

Repertorium der höheren Mathematik. Zweite Auflage, erster Band, erste Hälfte, herausgegeben von P. EPSTEIN. xv+527 pp. M. 10. Zweiter Band, erste Hälfte, herausgegeben von H. E. TIMERDING. xvi+534 pp. M. 10. Leipzig, B. G. Teubner, 1910.

THIS work is called a second edition of Pascal's Repertorium of Higher Mathematics.* It is, however, in many respects, a new work. The text has been thoroughly revised and greatly augmented. The half which has already appeared is nearly equal in size to the whole of the first edition. We are also informed (volume I, page viii) of a change in the purpose of the work. The aim of this edition is to give the reader, "a systematic survey of the total field of mathematics, based on genuine understanding."

As a further conspicuous departure from the original edition, which was the work of a single author, the editors have secured, as authors of the individual chapters, men particularly interested in the subjects covered by them. In the part of the work that has appeared, the authors are: in analysis, Hans Hahn, Alfred Loewy, H. E. Timerding, Paul Epstein; in geometry, J. Mollerup, H. Liebmann, H. Timerding, L. Heffter, G. Guareschi, M. Dehn, F. Dingeldey, L. Berzolari, G. Giraud, E. Ciani, H. Wieleitner. It is interesting

* A review of the first volume of the original Italian edition will be found in this BULLETIN, vol. 5 (1899), pp. 357–362.

to notice that, for the volume on geometry, the editors have secured several Danish and Italian authors. "The decrease of interest in geometry in Germany," we are told (volume II, page vi), "made it seem impossible to do otherwise."

In method of presentation, the authors have followed the first edition quite closely. They have endeavored to present in the clearest and concisest manner possible the principal theorems, formulas, definitions, and concepts of each of the subjects treated. Proofs are usually suppressed, but are sometimes indicated briefly. In references to the literature this work is a great improvement on its predecessor. The references are now always ample for further study of the main topic under discussion, but they will even yet, in some cases, be found inadequate by the reader whose interest is temporarily centered on an individual theorem.

The choice of subject matter is usually to be commended. Increased emphasis is laid, in the first volume, on the algebraic-group-theoretic chapters, and, in the second, on the chapters on the foundations of geometry. Other chapters, as, for example, those on differential and integral calculus and the calculus of finite differences, are given about the same space as in the first edition. The chapters on the calculus of probabilities and mathematical instruments have been omitted.

Related topics are in some cases insufficiently correlated. As an instance, we may cite the discussion of invariants of ternary cubics and quartics, in volume I, with the discussion of cubic and quartic curves in volume II. This inadequate correlation leads, at times, to unnecessary repetitions. A wider use of cross references would have been helpful.

The statements made are usually clear, concise, and accurate. There are occasional ambiguities, such as the failure to indicate clearly, in volume II, page 398, lines 15–27, whether the statements made are true for all quartics, or only for non-singular ones. There are also some errors. In the last theorem on page 158 of volume I, the $n + 1$ given functions should be of the same degree. The theorems in volume II, page 381, lines 11–14, and page 385, lines 27–30, are false. The theorem stated in volume II, page 406, lines 25–27, fails if the net contains precisely one or two double lines. The correct statement of this theorem should not have been omitted from Chapter XVII.

The publication of the rest of this Repertorium has been delayed far beyond the time originally set. Its appearance will be awaited with interest by mathematicians who have learned the usefulness of the part already published.

C. H. SISAM.

Elements of the Differential and Integral Calculus. By Professor A. E. H. LOVE. Cambridge University Press, 1909. 207 pp.

STARTING with the thesis that "the principles of the differential and integral calculus ought to be counted as a part of the intellectual heritage of every educated man or woman in the twentieth century, no less than the Copernican system or the Darwinian theory," Professor Love has put together in the eleven chapters of this small volume, the fundamental notions of analytic geometry and of the calculus, as well as some of their applications. The whole has been presented in such a way as not to require even as much knowledge of algebra and trigonometry as might reasonably be expected of a sophomore in an American college. In fact, Chapter VII, on Trigonometric functions, begins with an exposition of radian measurement, followed by a definition of sine and cosine. A review of the laws of indices and the interpretation of negative and fractional exponents precedes the discussion of the derivatives of general powers. Needless to say that no attempt is made to give a proof of the existence of a limit of $(1 + 1/n)^n$ as n increases indefinitely. By computing the function for a series of rapidly increasing values of n, the existence of the limit is made plausible.

An appendix of 25 pages is devoted to the discussion of the graph of the linear function, limits, indices and logarithms, the exponential limit, the mensuration of the circle and radian measure, trigonometric limits, and mechanical units. In this way the more ambitious reader of the book is given an opportunity to get a more rigorous treatment of some of the subjects treated more superficially in the text. The method for the calculation of e given in the appendix seems unnecessarily long to the reviewer.

The style of the book is very clear and it would seem that anybody of medium intelligence ought to be able to understand the leading principles of the calculus by a perusal of this text. The illustrations and applications are rather

monotonous and of the stereotyped character, such as the falling body, oscillation in a resisting medium; a greater variety of illustrations would doubtless make the book more attractive to the non-mathematical reader for whom it is chiefly intended.

It is an interesting fact that the need for books of this type is becoming more and more clearly recognized; it seems to be a part of the movement for the popularization of the principal results of the sciences; how far this can be carried in the case of mathematics seems doubtful, because mathematics is and must remain, as soon as we pass beyond the elementary stages, a highly theoretical subject, which does not lend itself to popularization. Quite a different thing it is of course to present some topic in mathematics in such a way as to make it available to those who have had at least some preparation; such as for instance giving undergraduate students an idea of what lies beyond or below, as has been done in J. W. A. Young's Monographs on Modern Mathematics and in J. W. Young's Fundamental Concepts of Algebra and Geometry; or again, to give to students of physics and chemistry a knowledge of those parts of mathematics that are essential to a full understanding of their own subjects. It is as a contribution to those fields that Professor Love's book is perhaps most valuable, taking its place by the side of such books as Nernst und Schoenflies's Mathematische Behandlung der Naturwissenschaften.

ARNOLD DRESDEN.

NOTES.

THE March number (volume 14, number 3) of the *Annals of Mathematics* contains the following papers: "Groups which contain an abelian subgroup of prime index," by G. A. MILLER; "On infinite systems of linear integral equations," by L. BRAND; "The method of monodromie with applications to three-parameter quartic equations," by R. P. BAKER; "Note on the existence theorem of a minimum of $\int_{xy}^{x'y'} Pdx + Qdy$," by E. SWIFT; "Continuant expressions for $\sqrt{a^2+b^2}$ and $(\sqrt{a^2+b^2}+a)^n$," by L. H. RICE.

AT the meeting of the London mathematical society held on February 13 the following papers were read: By T. C. LEWIS, "Figures in n-dimensional space analogous to orthocentric tetrahedra"; by J. E. LITTLEWOOD, "A property of the zeta function"; by G. H. HARDY, "The summability of Fourier's series"; by G. H. HARDY and J. E. LITTLEWOOD, "Trigonometric series which converge nowhere or almost nowhere"; by H. BOHR, "A theorem concerning power series"; by P. J. HEAWOOD, "The theorem of quadratic reciprocity"; by J. B. HOLT, "The irreducibility of Legendre's polynomials"; by W. H. YOUNG, "The mode of oscillation of a Fourier series and its allied series"; by H. T. H. PIAGGIO, "Some non-primary perpetuant syzygies of the second kind."

THE opening number of *Die Naturwissenschaften*, a weekly scientific journal edited by Dr. A. BERLINER and Dr. C. THESING in Berlin, appeared January 3, 1913. It proposes to furnish a general report of progress in pure and applied science, including mathematics and technical sciences. The present number contains an outline of the German report to the International commission on the teaching of mathematics by Professor A. GUTZMER, and a number of other papers on mathematical subjects are to follow. The subscription price is 24 marks per year.

UNIVERSITY OF CHICAGO. The following advanced courses in mathematics are announced for the summer quarter. All courses are four hours a week. By Professor E. H. MOORE: Fourier series, first term; Lemmas on the theory of point sets, first term.—By Professor O. BOLZA: Linear integral equations; Theory of functions.—By Professor F. R. MOULTON: Modern theories of analytic differential equations.—By Professor J. W. A. YOUNG: Critical review of secondary mathematics for teachers.—By Professor E. J. WILCZYNSKI: Theory of equations.—By Professor W. D. MACMILLAN: Observational astronomy; Celestial mechanics.—By Professor H. E. SLAUGHT: Differential equations.—By Professor A. C. LUNN: Graphical analysis; statistical mechanics.—By Professor G. A. BLISS: Projective geometry.

UNIVERSITY OF PENNSYLVANIA. The following courses are announced for the academic year 1913–1914: By Pro-

fessor E. S. CRAWLEY: Higher plane curves, three hours.—By Professor G. E. FISHER: Differential equations, three hours; Theory of functions of a complex variable, three hours.—By Professor I. J. SCHWATT: Definite integrals, three hours.—By Professor G. H. HALLETT: Theory of abstract groups, three hours; Introduction to higher algebra, three hours.—By Professor F. H. SAFFORD: Mathematical theory of elasticity, three hours; Partial differential equations, three hours.—By Professor M. J. BABB: History of mathematics, two hours; Theory of statistics, two hours.—By Professor G. G. CHAMBERS: Synthetic projective geometry (second course), three hours.—By Professor O. E. GLENN: Theory of invariants, three hours.—By Dr. H. H. MITCHELL: Theory of numbers, three hours.—By Dr. R. L. MOORE: Theory of point sets, with applications, three hours.—By Dr. F. W. BEAL: Differential geometry, three hours.

THE following courses in mathematics are announced for the summer semester, 1913.

UNIVERSITY OF BONN.—By Professor E. STUDY: Differential geometry, I, two hours; Higher geometry, I, two hours; Seminar, two hours.—By Professor F. LONDON: Introduction to the theory of differential equations, three hours; Axonometry and perspective, three hours.—By Professor F. HAUSDORFF: Elements of differential and integral calculus, with exercise, five hours.—By DR. J. O. MÜLLER: Algebraic equations, three hours; Introduction to geodesy, two hours; Proseminar, two hours.

UNIVERSITY OF GÖTTINGEN.—By Professor H. HILBERT: Elements and principles of mathematics, four hours; Theory of motion of electrons, two hours; Seminar, two hours.—By Professor C. RUNGE: Differential and integral calculus, I, six hours; Seminar, two hours.—By Professor F. BERNSTEIN: Theory of probabilities and applications, four hours; Mathematics of insurance, two hours; Seminar, two hours.—By Professor O. TOEPLITZ: Differential equations, four hours.—By Dr. H. WEYL: Algebra, four hours; Survey of the theory of elliptic, abelian, and automorphic functions, two hours.—By Dr. L. v. SENDEN: Descriptive geometry, four hours; Graphical methods of applied mathematics, six hours.—By Dr. HECKE: Definite integrals, with applications, three hours;

Theory of algebraic fields, four hours.—By Dr. R. COURANT: Calculus of variations, three hours; Exercises in analytic geometry of space, two hours.—By Professor L. PRANDTL: Mechanics of continua, three hours; Seminar in aeronautics, two hours.

UNIVERSITY OF MUNICH.—By Professor F. LINDEMANN: Analytic geometry of space, five hours; Theory of definite integrals and of Fourier series, four hours; Mechanics of deformable bodies, two hours; Seminar, two hours.—By Professor A. VOSS: Integral calculus, four hours; Introduction to the theory of algebraic curves and surfaces, three hours.—By Professor A. PRINGSHEIM: Algebra, II, four hours; Selected chapters in the theory of functions, four hours.—By Professor G. HARTOGS: Descriptive geometry, with exercises, seven hours.—By Professor BRUNN: Analysis situs, three hours.—By Dr. ROSENTHAL: Algebraic analysis, four hours; Theory of sets of points, three hours.—By Dr. DINGLER: Elements of differential geometry of plane curves, three hours.—By Dr. BOEHM: Advanced integral calculus, and boundary problems, three hours; Exercises in mathematics of insurance, two hours; Mathematical statistics, two hours.

DURING the last thirty-six years Harvard University has sent students of mathematics to Europe on travelling fellowships in almost unbroken succession. In only two of these years has there been no student of mathematics or mathematical physics studying abroad on such a fellowship, while several times there have been two or three such appointments simultaneously. These opportunities for foreign study were about doubled three years ago by the Sheldon bequest of $350,000, the income of which is used in grants for this purpose (not necessarily in mathematics), so that it may be expected that in future, as during the past four years, there will ordinarily be at least two travelling fellows from Harvard in mathematics each year with stipends varying from $500 to $1150. The first Sheldon Fellow in mathematics was Dr. G. C. EVANS who held the appointment for two years while studying in Rome. This year Mr. W. C. GRAUSTEIN is Sheldon Fellow at Bonn, while Mr. E. S. ALLEN in Rome and Dr. TOMLINSON FORT in Göttingen also hold travelling fellowships from Harvard for the study of mathematics. The ap-

pointments are invariably given to men who have distinguished themselves by their work as resident students at Harvard, though frequently to students whose bachelor's degree was taken elsewhere.

THE Lobachevsky prize of the academy of sciences of Kasan has been awarded to Professor F. SCHUR, of the University of Strassburg, for his researches in the foundations of geometry.

As a memorial to Professor P. G. TAIT, it is proposed to establish a second chair of natural philosophy in the University of Edinburgh. A committee has been organized to raise a fund of £20,000 to £25,000 for this purpose. Subscriptions should be sent to the honorary treasurer, Sir G. M. Paul, 16 St. Andrew Square, Edinburgh.

MR. R. H. MOODY has been appointed professor of mathematics at Muir's Central College, Allahabad.

PROFESSOR FELIX KLEIN, of the University of Göttingen, has been relieved permanently of his university duties, on account of ill health.

DR. P. BERNAYS has been appointed docent in mathematics at the University of Zürich.

DR. A. HOBORSKI has been appointed docent in mathematics at the University of Cracow.

PROFESSOR H. ANDOYER has been appointed professor of mathematical astronomy at the University of Paris.

PROFESSOR C. CARATHÉODORY, of the technical school at Breslau, has accepted a professorship of mathematics at the University of Göttingen.

PROFESSORS M. DISTELI and K. HEUN, of the technical school at Carlsruhe, have received the title of Hofrat.

PROFESSOR E. STUDY, of the University of Bonn, has received the title of privy councillor.

DR. M. STUYVAERT has been appointed docent in algebraic analysis at the University of Ghent.

DR. H. MOHRMANN, of the technical school at Carlsruhe, has accepted the professorship of mathematics at the mining academy of Clausthal.

DR. N. E. NÖRLUND has been appointed professor of mathematics at the University of Lund.

DR. L. A. H. WARREN has been appointed professor of mathematics at the University of Manitoba.

PROFESSOR MAXIME BÔCHER, of Harvard University, has been appointed exchange professor to lecture at the Sorbonne during the first half of the academic year 1913–1914.

PROFESSOR O. P. AKERS, of Allegheny College, has been granted leave of absence for the academic year 1913–1914, to travel and study in Europe.

DR. J. E. MANCHESTER, of the University of Minnesota, died January 24, at the age of 57 years.

BOOK catalogues: R. Friedlander, Karlstrasse 11, Berlin, catalogue 481–482, 94 and 14 pages.—A. Hermann, 6 Rue de la Sorbonne, Paris, mathematics, physics, and natural science, 98 pages.—A. Reichmann, Hauptstrasse 18, Vienna, catalogue 79, 1653 titles.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BAIRE (R.). Théorie des nombres irrationnels, des limites et de la continuité. 2e édition. Paris, Vuibert, 1912. 8vo. 64 pp.

BERNSTEIN (S.). On the best approximation to continuous functions by means of polynomials of given degree. (Russian.) Charkow, 1912. 8vo. 153 pp. $0.75

BEST (L.). Elementarer Beweis des Fermatschen Satzes. Darmstadt, Schlapp, 1912. 8vo. 4 pp. M. 0.50

BIDDLE (A. D.). Constructive theory of the unicursal plane quartic by synthetic methods. (Publications of the University of California.) Berkeley, 1912. 8vo. 28 pp. $0.60

BRUNS (H.). Von Ptolemäus bis Newton. (Rektoratsrede.) Leipzig, Edelmann, 1912. 8vo. 25 pp. M. 0.75

BÜCHER, Neue, über Naturwissenschaften und Mathematik. Mitgeteilt Herbst, 1912. Leipzig, Hinrichs. 8vo. Pp. 47–70. M. 0.30

BURALI-FORTI (C.) e MARCOLONGO (R.). Analyse vectorielle générale. Volume 1: Transformations linéaires. Traduit de l'italien par P. Baridon. Pavia, Mattei, 1912. 8vo. 19 + 179 pp. L. 6.00

CALVET-AZAL. Essai sur la notion de quantité imaginaire. Nouvelle édition. Paris, Gauthier-Villars, 1912. 8vo. 60 pp.

FIEDLER (R.). See JORDAN (C.).

FRIZELL (A. B.). Foundations of arithmetic. (Scientific Bulletin of the University of Kansas.) Lawrence, 1911. 8vo. 31 pp. $0.50

GALÁN (G.). See PESCI (G.).

GAUSS (C. F.). Fragmente zur Theorie des arithmetisch-geometrischen Mittels aus den Jahren 1797–1799. (Materialien für eine wissenschaftliche Biographie von Gauss. 2tes Heft.) Leipzig, Teubner, 1912. 8vo.

GHERSI (I.). Matematica dilettevole e curiosa: problemi bizarri, paradossi algebrici, geometrici e meccanici, moto perpetuo, grandi numeri, curve e loro tracciamento meccanico, ecc. Milano, Hoepli, 1913. 24mo. 8 + 730 pp. L. 9.50

GRASSMANN (H.). Projektive Geometrie der Ebene, unter Benutzung der Punktrechnung dargestellt. 2ter Band: Ternäres. Leipzig, Teubner, 1912. 8vo. 12 + 410 pp. Cloth. M. 19.00

GRÜBEL (A.). Ueber lineare Differentialgleichungen mit bekanntem linearen Multiplikator. (Diss.) Basel, 1912. 8vo. 57 pp.

HEIBERG (G. H.). See HERO ALEXANDRINUS.

HERO ALEXANDRINUS. Opera quae supersunt omnia. Volumen IV: Definitiones cum variis collectionibus quae feruntur Geometrica. Copiis G. Schmidt usus edidit G. L. Heiberg. (Graece et Germanice.) Leipzig, 1912. 8vo. 29 + 443 pp. M. 9.00

HOFSTETTER (P.). Die Bernoullische Funktion und die Gammafunktion. (Diss.) Bern, 1911. 8vo. 109 pp. M. 3.00

JORDAN (C.) et FIEDLER (R.). Contribution à l'étude des courbes convexes fermées et de certaines courbes qui s'y rattachent. Paris, Hermann, 1912. 8vo. 81 pp.

KEMPNER (A. J.). Ueber das Waringsche Problem und einige Verallgemeinerungen. (Diss.) Göttingen, 1912. 8vo. 56 pp.

KOWALEWSKI (G.). Einführung in die Infinitesimalrechnung mit einer historischen Uebersicht. 2te, völlig umgearbeitete Auflage. (Aus Natur und Geisteswelt. Neue Auflage. Nr. 197.) Leipzig, Teubner, 1913. 8vo. 6 + 106 pp. Cloth. M. 1.25

LINDEMANN (F. und L.). See PICARD (E.).

LINDOW (M.). Differential- und Integralrechnung mit Berücksichtigung der praktischen Anwendung in der Technik. (Aus Natur und Geisteswelt. Nr. 387.) Leipzig, Teubner, 1913. 8vo. 7 + 111 pp. Cloth. M. 1.25

LOKOTSCH (K.). Avicenna als Mathematiker, besonders die planimetrischen Bücher seiner Euklidübersetzung. (Diss.) Bonn, 1912. 8vo. 69 pp.

LÖWENKLAU (L.). Zum grossen Fermatschen Satz. Dresden, Köhler, 1912. 8vo. 4 pp. M. 0.50

MARCOLONGO (R.). See BURALI-FORTI (C.).

NEWSON (H. B.). Theory of collineations. (Scientific Bulletin of the University of Kansas.) Lawrence, 1911. 8vo. 10 + 319 pp. $2.50

OSGOOD (W. F.). Lehrbuch der Funktionentheorie. 1ter Band. 2te Auflage. (Teubner's Sammlung. Neue Auflage. Band XX.) Leipzig, Teubner, 1912. 8vo. 12 + 766 pp. Cloth. M. 18.00

PERRON (O.). Die Lehre von den Kettenbrüchen. (Teubner's Sammlung. Band 36.) Leipzig, Teubner, 1913. 8vo. 12 + 520 pp. Cloth. M. 22.00

PESCI (G.). Las tablas gráficas de Luyando. Contribucion a la historia de la nomografia. Traduccion de G. Galán. Zaragoza, 1909. 8vo. 15 pp.

PICARD (E.). Das Wissen der Gegenwart in Mathematik und Naturwissenschaft. Deutsch mit erläuternden Anmerkungen von F. und L. Lindemann. (Wissenschaft und Hypothese, XVI.) Leipzig, Teubner, 1913. 8vo. 4 + 292 pp. Cloth. M. 6.00

SCHLESINGER (L.). Ueber Gauss' Arbeiten zur Funktionentheorie. (Materialien für eine wissenschaftliche Biographie von Gauss. 3tes Heft.) Leipzig, Teubner, 1912. 8vo.

SCHMIDT (G.). See HERO ALEXANDRINUS.

SMITH (E. R.). Zur Theorie der Heineschen Reihe und ihrer Verallgemeinerung. (Diss.) München, 1911. 8vo. 63 pp.

STOHLER (A.). Multiplikation und Teilung von Sigmaquotienten. (Diss.) Basel, 1911. 8vo. 87 pp.

THIJWISSEN (A. J. H.). Het vraagstuk van Dirichlet. Leiden, 1911. 8vo. 11 + 191 pp.

VALENTINER (S.). Vektoranalysis. 2te, umgearbeitete Auflage. (Sammlung Göschen. Neue Auflage. Nr. 354.) Berlin, Göschen, 1912. 8vo. 156 pp. Cloth. M. 0.80

WEBER (H.). Lehrbuch der Algebra. Kleine Ausgabe in 1 Bande. Braunschweig, Vieweg, 1912. 8vo. 10 + 528 pp. M. 15.00

WEICKMANN (L.). Beiträge zur Theorie der Flächen mit einer Schar von Minimalgeraden. (Diss.) München, 1912. 8vo. 51 pp.

ZEUTHEN (G. H.). Die Mathematik im Altertum und im Mittelalter. (Die Kultur der Gegenwart. 3ter Teil, 1te Abteilung, 1te Lieferung.) Leipzig, Teubner, 1912. 8vo. 5 + 95 pp. M. 3.00

II. ELEMENTARY MATHEMATICS.

AMALDI (U.). See ENRIQUES (F.).

BARNARD (S.) and CHILD (J. M.). A new geometry. Parts 1 and 2. London, Macmillan, 1912. 8vo. 334 pp.

BAUER (W.) und HANXLEDEN (E.). Lehrbuch der Mathematik. 2ter Band: Planimetrie, Stereometrie und Trigonometrie. Braunschweig, Vieweg, 1912. 8vo. 8 + 179 pp. Cloth. M. 2.40

BOUASSE (H.) et TURRIÈRE (E.). Exercices et compléments de mathématiques générales. Paris, Delagrave, 1912. 8vo. 15 + 504 pp. Fr. 18.00

BOUCHENY (G.) et GUÉRINET (A.). L'algèbre au cours complémentaire. 2e édition. Paris, Larousse, 1912. 8vo. 132 pp. Fr. 1.40

——. La géométrie au cours complémentaire. 2e édition. Paris, Larousse, 1912. 8vo. 336 pp. Fr. 2.80

BOURLET (C.). Eléments d'algèbre. Classes de 3e A, 2e et 1re A et B. 4e édition, revisée. Paris, Hachette, 1912. 16mo. 363 pp. Fr. 3.00

CHAMBRÉ (A.). See KOCH (W.).

CHILD (J. M.). See BARNARD (S.).

CRANTZ (P.). Aufgaben aus der Trigonometrie, der Stereometrie und der analytischen Geometrie. Leipzig, Teubner, 1913. 8vo. 4+94 pp. M. 1.40

EIBE (T.). See EUKLID.

ELÉMENTS de trigonométrie rectiligne. Par F. J. Paris, Gigord, 1912. 16mo. 7 + 256 pp.

ENRIQUES (F.) e AMALDI (U.). Elementi di geometria. 3a edizione. Bologna, Zanichelli, 1912. 16mo. 8+272 pp. L. 2.00

——. Nozioni di geometria. 2a edizione. Bologna, Zanichelli, 1912. 16mo. 7+195 pp. L. 1.80

EUKLID. Elementer, VII–XIII. Oversat af T. Eibe. Kjöbenhavn, 1912. 8vo. 486 pp. Kr. 9.00

FRONTERA (D. J.). See SONNET (H.).

FUSS (K.). Die wichtigsten Sätze der Planimetrie, Stereometrie und Trigonometrie. 4te, vermehrte Auflage. Nürnberg, Korn, 1912. 8vo. 8+249 pp. Cloth. M. 3.00

GUÉRINET (A.). See BOUCHENY (G.).

HÄDICKE (G.). Einführung in die neuere Geometrie. Ein Vorschlag zur Reform des elementar-geometrischen Unterrichts. 1ter Teil: Symmetrie und Kongruenz. Berlin, Oehmigke, 1912. 8vo. 26 + 241 pp. Cloth. M. 3.60

HANXLEDEN (E.). See BAUER (W.).

HEEGAARD (P.). Der Mathematikunterricht in Dänemark. (Internationale mathematische Unterrichtskommission.) Kjöbenhavn, Gyldendalske Boghandel, 1912.

KISSELEW (A.). Elementare Geometrie. Uebersetzt von R. v. Zeddelmann. Riga, Löffler, 1912. 8vo. 3+329 pp. M. 3.50

KLEEFSTRA (J.). Leerboek der Meetkunde. 3 delen. Groningen, 1911. 8vo. 115, 94 and 78 pp. M. 5.50

KOCH (W.) und CHAMBRÉ (A.). Graphische Algebra. Stuttgart, Grub, 1913. 8vo. 24 pp. M. 1.30

LAAGER (F.). Planimetrische Konstruktionsaufgaben. 2te Auflage. Zürich, Speidel, 1912. 8vo. 47 pp. M. 1.20

LACKEMANN. Elemente der Geometrie. Bearbeitet von R. Kreuschmer, 1ter Teil: Planimetrie. 10te, verbesserte Auflage. Breslau, Hirt. 1912. 8vo. 148 pp. M. 2.00

LANE (F. O. and J. A. C.). A special algebra. London, Arnold, 1912. 8vo. 296 pp. 3s. 6d.

LEHMANN (H.). Die technischen Hilfsmittel für den physikalischen und mathematischen Unterricht am König Georg-Gymnasium zu Dresden. Dresden, 1912. 4to. 48 pp. M. 2.50

MÖHLE (F.). Der mathematische und naturwissenschaftliche Unterricht an den preussischen Lyzeen nach der Neuordnung von 1908. (Schriften des deutschen Ausschusses. 15tes Heft.) Leipzig, Teubner, 1913. 8vo. 4+48 pp. M. 1.50

PAOLETTI (G.). Elementi di geometria. Rocca S. Casciano, Cappelli, 1912. 16mo. 206 pp. L. 2.00

PINCHERLE (S.). Lezioni di algebra elementare. Bologna, Zanichelli, 1912. 8vo. 5+389 pp. L. 4.50

REPORT of the American commissioners of the international commission on the teaching of mathematics. Washington, United States Bureau of Education, 1912.

SCHMALZ (K.). Schul-Mathematik in Tabellen, Dispositionen und Repetitorium. Halle, Waisenhaus, 1912. 8vo. 36 pp. M. 0.75

SIGISMONDO (F.). Lezioni d'algebra elementare. Rimini, Capelli, 1913. 16mo. 120 pp. L. 1.50

SONNET (H.). Primeros elementos de geometria. Traducidos por D. J. Frontera. Nova edición. Paris, Hachette, 1912. 16mo. 239 pp. Fr. 3.00

—— et FRONTERA (G.). Eléments de géométrie analytique. 12e édition. Paris, Hachette, 1912. 8vo. 758 pp. Fr. 8.00

TESTI (G. M.). Compendio di algebra elementare. Livorno, Giusti, 1912. 16mo. 9+140 pp. L. 1.00

TURRIÈRE (E.). See BOUASSE (H.).

VASNIER. Cours de géométrie. 1re partie: Géométrie plane. Paris, Union typographique, 1911. 8vo. 406 pp.

ZEDDELMANN (R. v.). See KISSELEW (A.).

ZIMMERMANN (H.). Rechentafel nebst Sammlung häufig gebrauchter Zahlenwerte. 7te Auflage. Ausgabe B. Berlin, Ernst, 1913. 8vo. 34+204+20 pp. Cloth. M. 6.00

III. APPLIED MATHEMATICS.

ADAMI (F.). Die Elektrizität. 2 Teile. (Bücher der Naturwissenschaft. 14ter Band.) Leipzig, Reclam, 1912. 16mo. 180 pp. Cloth. M. 1.50

APPELL (P.) et CHAPPUIS (J.). Leçons de mécanique élementaire. 3e édition, entièrement refondue. Paris, Gauthier-Villars, 1913. 16mo. 9+416 pp. Fr. 6.00

BERNIOLLE (P.). Leçons de géométrie descriptive. Classes de 1re C et D. 4e édition. Paris, Paulin, 1912. 18mo. 194 pp.

BJERKNES (V.). Dynamische Meteorologie und Hydrographie. 1ter Teil: Statik der Atmosphäre und der Hydrosphäre. Von V. Bjerknes und J. W. Sandström. Deutsche Uebersetzung von F. Kirchner. Braunschweig, Vieweg, 1912. 6+126+36+30+22 pp. M. 36.00

BLANCARNOUX (P.). Toute la mécanique rationnelle et appliquée a la portée de tous. Tomes 11 et 12. Paris, Geisler, 1912. 8vo. 84 and 136 pp.

BLESSING (G. F.) and DARLING (L. A.). Elements of descriptive geometry. New York, Wiley, 1912. 8vo. 14+219 pp. $1.50

BRILL (A.). Das Relativitätsprinzip. Eine Einführung in die Theorie. Leipzig, Teubner, 1912. 8vo. 3+28 pp. M. 1.20

BUSCH (H.). Stabilität, Labilität und Pendelungen in der Elektrotechnik. 1ter Teil. (Diss.) Göttingen, 1911. 8vo. 144 pp.

CHAPPUIS (J.). See APPELL (P.).

CHASSAGNY (M.). Manuel théorique et pratique d'électricité. 4e édition. Paris, Hachette, 1912. 16mo. 507 pp. Fr. 4.00

COSSERAT (E. et F.). See PERRY (J.).

DARLING (L. A.). See BLESSING (G. F.).

DAVAUX (E.). See PERRY (J.).

DORGUEIL (S.). Tables aéronautiques. Paris, Libraire aéronautique, 1912. 8vo. 32 pp. Fr. 3.00

ELÉMENTS de géométrie descriptive. Par F. J. Paris, Gigord, 1913. 12mo. 462 pp.

ENDERLE (A.). Zur Theorie der elektrischen und magnetischen Doppelbrechung von Flüssigkeiten. (Diss.) Freiburg, 1912. 8vo. 85 pp.

ENZYKLOPÄDIE der mathematischen Wissenschaften. Band VI 1 B: Geodäsie und Geophysik. 3tes Heft: Dynamische Meteorologie, von Exner und Trabert. Leipzig, Teubner, 1912. 8vo. Pp. 179–234. M. 1.80

FÖRSTER (G.). See REINHERTZ (C.).

GANDILLOT (M.). Abrégé sur l'helice et la résistance de l'air. Paris, Gauthier-Villars, 1912. 4to. 192 pp. Fr. 10.00

KIRCHNER (F.). See BJERKNES (V.).

LARMINAT (E. DE). Topographie pratique de reconnaissance et d'exploration suivie de notions élémentaires pratiques de géodésie et d'astronomie de campagne. 3e édition. Paris, Charles-Lavauzelle, 1912. 8vo. 404 pp. Fr. 10.00

LEICK (W.). Astronomische Ortsbestimmungen mit besonderer Berücksichtigung der Luftschiffahrt. Leipzig, 1912. 8vo. 8+130 pp. M. 2.80

MARCHAND BEY (E. E.). Réflexions de mécanique générale sur l'univers et la matière. Réfutation mathématique des lois de Newton. Nouveautés scientifiques. Chatou, 1912. 8vo. 78 pp.

MARSH (H. W.). Industrial mathematics. New York, Wiley, 1912. 8vo. 8+476 pp. $2.00

MARTIN (L. A., JR.). Textbook of mechanics. Volume 4: Applied statics. New York, Wiley, 1912. 12mo. 12+198 pp. Cloth. $1.50

MAYHER (W.). Die astronomische Zeitrechnung der Völker von ihrem Ursprung bis zur Gegenwart und die Einheitszeit. Mannheim, Haas, 1912. 8vo. 115 pp. Cloth. M. 4.00

MERRIMAN (M.) and others. American civil engineer's pocketbook. 2d edition, enlarged. New York, Wiley, 1912. 12mo. 8+1473 pp. $5.00

MEWES (L.). Definitive Bahnbestimmung des Kometen 1846 VII. (Diss.) Breslau, 1912. 8vo. 57 pp.

MIDDLETON (G. A. T.). Surveying and surveying instruments. 3d edition, enlarged. London, 1912. 8vo. 184 pp. Cloth. 5s. 6d.

MONTANARI (C.). Elementi di geometria descrittiva. 3a edizione, riveduta e corretta. Livorno, Giusti, 1913. 16mo. 40 pp. L. 0.50

MOROSOW (N.). Functions. A concrete presentation of the calculus. Applications to physics and geometry. (Russian.) Kieff, 1912. 8vo. 464 pp. $2.00

MURRAY (J. E.). A handbook of wireless telegraphy. Its theory and practice. 4th edition, revised and enlarged. London, Lockwood, 1912. 8vo. 458 pp. 10s. 6d.

NEKRASSOW (P. A.). Theory of probabilities. 2d edition. (Russian.) Petersburg, 1912. 8vo. 532 pp. $3.00

NERNST (W.). Theoretische Chemie vom Standpunkte der Avogadroschen Regel und der Thermodynamik. 7te Auflage. Stuttgart, Enke, 1913. 8vo. 16+838 pp. Half leather. M. 24.80

PALMER (A. D.). The theory of measurements. New York, McGraw-Hill, 1912. 248 pp. $2.50

PERRY (J.). Mécanique appliquée. Traduit sur ia 9e édition anglaise, par E. Davaux. Avec des additions et un appendice sur la mécanique des corps déformables, par E. et F. Cosserat. Tome 1: L'énergie mécanique. Paris, Hermann, 1913. 8vo. 5+405 pp.

PLANCK (M.). Das Prinzip der Erhaltung der Energie. (Preisgekröntes Schrift.) 3te Auflage. (Wissenschaft und Hypothese. Neue Auflage. VI.) Leipzig, Teubner, 1913. 8vo. 16+278 pp. Cloth. M. 6.00

——. Vorlesungen über die Theorie der Wärmestrahlung. 2te, teilweise umgearbeitete Auflage. Leipzig, Barth, 1913. 8vo. 12+206 pp. M. 7.80

REINHERTZ (C.). Geodäsie. 2te Auflage, neubearbeitet von G. Förster. (Sammlung Göschen. Neue Auflage. Nr. 102.) Berlin, Göschen, 1912. 8vo. 169 pp. Cloth. M. 0.80

RIEFLER (S.). Tabellen der Luftgewichte J^b_t, der Druckäquivalente β^b_t, und der Gravitation g. Berlin, Springer, 1912. 8vo. 4+93 pp. Cloth. M. 6.00

RINCKLAKE (A.). Astrochemie und -mechanik. Berlin, 1912. 8vo. M. 1.00

SANDSTRÖM (J. W.). See BJERKNES (V.).

SCHAEFER (K. L.). Musikalische Akustik. 2te, neubearbeitete Auflage. (Sammlung Göschen. Neue Auflage. Nr. 21.) Berlin, Göschen, 1912. 8vo. 144 pp. Cloth. M. 0.80

VEITHEN (C.). Ueber die Verwendung der Rechenmaschine bei der Bahnbestimmung von Planeten. (Diss.) Leipzig, 1912. 4to. 40 pp.

VOGLER (C. A.). Geodätische Uebungen für Landmesser und Ingenieure. 3te Auflage. Teil II: Winterübungen. Berlin, 1913. 8vo. 8+204 pp. Cloth. M. 7.00

VUIBERT (H.). Les anaglyphes géométriques. Géométrie stéréoscopique. Paris, 1912. 8vo. Fr. 2.00